Self-organised criticality in random graph processes

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original. This thesis is submitted to the University of Oxford for the degree Doctor of Philosophy, and has not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

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Abstract

In the first half of this thesis, we study the random forest obtained by conditioning the Erdős–Rényi random graph $G(N, p)$ to include no cycles. We focus on the *critical window*, in which $p(N) = \frac{1+\lambda N^{-1/3}}{N}$, as studied by Aldous for $G(N, p)$. We describe a scaling limit for the sizes of the largest trees in this critical random forest, in terms of the excursions above zero of a particular reflected diffusion. We proceed by showing convergence of the reflected exploration process associated to the critical random forests, using careful enumeration of classes of forests, and the asymptotic properties of uniform trees.

In the second half of this thesis, we study a random graph process where vertices have one of k types. An inhomogeneous random graph represents the initial connections between vertices, and over time new edges are added homogeneously, as in the classical random graph process. Each vertex is frozen at some rate, resulting in the removal of its entire component. This is a version of the frozen percolation model introduced by Ráth, which (under mild conditions) exhibits self-organised criticality: the dynamics first drive the system to a critical state, and from then on maintain it in criticality.

We prove a convergence result for the proportion of vertices of each type which survive until time t, and describe the local limit in terms of a multitype branching process whose parameters are critical and given by the solution to an unusual differential equation driven by Perron–Frobenius eigenvectors. The argument relies on a novel multitype exploration process, leading to a concentration result for the proportion of types in all large components of a near-critical inhomogeneous random graph; and on a stronger convergence result for mean-field frozen percolation, when the initial graphs may be random.

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Chapter 1

Introduction

In this short introduction, we introduce the models and techniques on which the rest of this thesis is based. In particular, we review the theory of the Erdős–Rényi random graph model, and some examples of random trees. A key method in studying these models is an *exploration process*, which allows us to reveal the structure of a random object one vertex at a time and track it in a Markovian manner. We introduce some natural examples of exploration processes, and explain how they can be used to describe asymptotic properties of the underlying graphs.

The second half of this thesis concerns graph-valued processes which exhibit *self-organised criticality*. We will review the history of such models, and introduce *mean-field frozen percolation*, an adaptation of which will be the main focus of Chapters 4 and 5.

1.1 Random graphs, random trees and coalescence

Throughout this thesis, a *graph* $G = (V, E)$ will always be undirected and simple. That is, G contains no loops nor multiple edges, so $E \subseteq V^{(2)}$, the set of unordered pairs of distinct vertices in V.

The *Erdős–Rényi random graph* has been one of the most widely studied random structures since its introduction in the 1950s. We define the random graph $G(N, p)$ as follows. The set of vertices is taken to be $[N] := \{1, \dots, N\}$, and then each potential edge ij (for $i < j$) is present with probability p, independently of all other edges. Erdős and Rényi [22, 23] initially studied $G(N, m)$, a uniformly-chosen graph with vertex set $[N]$ and precisely m edges. In one of the first and most famous examples of the *probabilistic method*, Erdős [21] used $G(N, 1/2)$ to derive lower bounds for Ramsey numbers.

The model $G(N, p)$ as defined in the previous paragraph was first introduced by Gilbert [30]. It is worth observing that for any $p \in (0, 1)$, the random graph $G(N, p)$ conditioned to contain exactly m edges has the same distribution as $G(N, m)$. For $p \leq p'$, there is a natural coupling of $G(N, p)$ and $G(N, p')$ such that

$$E(G(N, p)) \subseteq E(G(N, p')). \tag{1.1}$$

We can achieve this by, for example, taking $G(N, p)$ and (to borrow terminology from Bollobás [14]) *sprinkling* extra edges independently with probability $\frac{p'-p}{1-p}$ between any pair of vertices unconnected in $G(N, p)$. Alternatively, we can issue each potential edge $e \in [N]^{(2)}$ an independent random variable $U_e \sim U[0, 1]$. Then, we include e in $E(G(N, p))$ iff $U_e \leq p$. This gives a coupling of $G(N, p)$ for all $p \in [0, 1]$.

Definition 1.1. We will usually assume $p = p(N)$ is a function of N, and we will be interested in asymptotic properties of $G(N, p)$ as $N \to \infty$. We say that a property A holds *with high probability* (w.h.p.) in $G(N, p)$ if

$$\mathbb{P}(A \text{ holds in } G(N, p)) \to 1,$$

as $N \to \infty$.

Definition 1.2. We use the following notation to describe asymptotic scalings. As usual, if $(a_N), (b_N)$ are real-valued sequences, we write $a_N = O(b_N)$ if $\frac{a_N}{b_N}$ is bounded as $N \to \infty$, and $a_N = o(b_N)$ if $\frac{a_N}{b_N} \to 0$ as $N \to \infty$. Furthermore, when both sequences are positive, we write $a_N = \Theta(b_N)$ if

$$0 < \liminf_{N\to\infty} \frac{a_N}{b_N} \leq \limsup_{N\to\infty} \frac{a_N}{b_N} < \infty.$$

We also want to use similar language for random variables, but there is a risk of ambiguity. Following Janson [32] closely, we make the following pair of definitions.

Definition 1.3. Let (A_N) be a family of random variables, and (b_N) a positive real-valued sequence. Then we say $A_N = O_{\mathrm{p}}(b_N)$ if A_N/b_N is bounded in probability. That is,

$$\lim_{C\to\infty} \limsup_{N\to\infty} \mathbb{P}(|A_N| > Cb_N) = 0.$$

Similarly, we say $A_N = o_{\mathrm{p}}(b_N)$ if $\lim_{N\to\infty} \mathbb{P}(|A_N| > cb_N) = 0$ for all $c > 0$. Finally, we say $A_N = \Theta_{\mathrm{p}}(b_N)$ if

$$\lim_{C\to\infty} \limsup_{N\to\infty} \mathbb{P}(A_N > Cb_N) = \lim_{c\to\infty} \limsup_{N\to\infty} \mathbb{P}(A_N < cb_N) = 0.$$

Definition 1.4. Then we say $A_N = O(b_N)$ with high probability if for some $C > 0$ the property $\{|A_N| \le Cb_N\}$ holds with high probability. We also say that $A_N = o(b_N)$ with high probability if the properties $\{|A_N| \le cb_N\}$ hold with high probability for all $c > 0$. Similarly, we say that $A_N = \Theta(b_N)$ with high probability if for some $0 < c \le C$ the property $\{cb_N \le A_N \le Cb_N\}$ holds with high probability.

Note that $A_N = O(b_N)$ with high probability implies $A_N = O_{\mathrm{p}}(b_N)$, but the converse statement is false. However, $A_N = o_{\mathrm{p}}(b_N)$ is equivalent to $A_N = o(b_N)$ with high probability.

1.1.1 Erdős–Rényi process: sparse regime

In this thesis, we will mostly be concerned with $G(N,p)$ in the *sparse regime*, that is $p = \Theta(1/N)$. In this regime, the typical number of edges present in $G(N,p)$ is $\Theta(N)$, and the degree of a typical vertex is $\Theta(1)$. Indeed, for any vertex v in $G(N,c/N)$, the degree of v has the Binomial$(N-1,c/N)$ distribution, which is well-approximated by Poisson(c) for large N.

Motivated by the coupling (1.1), we consider a *random graph process* $\left(\mathcal{G}^N(t), t \ge 0\right)$ as follows. $\mathcal{G}^N$ takes values among the set of graphs on vertex set $[N]$. We set $\mathcal{G}^N(0)$ to be the empty graph, and add each potential edge in $[N]^{(2)}$ independently at rate $\frac{1}{N}$. It is easily seen that $\mathcal{G}^N$ is Markov, and satisfies

$$\mathcal{G}^N(t) \stackrel{d}{=} G(N, 1 - e^{-t/N}).$$

Note that for large N, we have $1 - e^{-t/N} = (1 + o(1))t/N$.

One of the most studied properties of the sparse random graph process is the emergence of the so-called *giant component*. Let $|\mathcal{C}_1^N| \geq |\mathcal{C}_2^N| \geq \dots$ be the sizes of the components of a graph in decreasing order. The size of the largest component $\mathcal{C}_1^N$ in $G(N, p)$ undergoes a phase transition at $p = \frac{1}{N}$. The regimes can be summarised as follows.

- For $p = \frac{c}{N}$, with $c < 1$, then $|\mathcal{C}_1^N| = O(\log N)$ as $N \to \infty$, with high probability. We say that such a random graph is *subcritical*.
- For $p = \frac{c}{N}$, with $c > 1$, then $|\mathcal{C}_1^N| = (\zeta_c + o(1))N$, where ζ_c is a positive constant, and $|\mathcal{C}_2^N| = O(\log N)$ with high probability. We say such a random graph is *supercritical* and that $\mathcal{C}_1^N$ is the (unique) *giant component*.
- For $p = \frac{1}{N}$, then $|\mathcal{C}_1^N|, \dots, |\mathcal{C}_k^N| = \Theta_{\mathrm{p}}\left(N^{2/3}\right)$ for any $k \in \mathbb{N}$. We say such a random graph is *critical*. As we shall see, it is possible to describe the distributional limit of $N^{-2/3}\left(|\mathcal{C}_1^N|, \dots, |\mathcal{C}_k^N|\right)$. Note that this does not assert that *all* components have size $\Theta\left(N^{2/3}\right)$.

The methods used by Erdős and Rényi in [23] mostly involve careful bounds on the proportion of vertices contained within small trees via second-moment methods. We will shortly introduce some more modern approaches, based on approximating the structure of $G(N, c/N)$ locally by a branching process.

1.1.2 Galton–Watson trees

We will shortly define a relevant family of branching processes. First, we define the Ulam–Harris notation for the space of trees on which they are supported.

Definition 1.5. Let $\mathcal{U} := \{\varnothing\} \cup \bigcup_{k \geq 1} \mathbb{N}^k$. For $u \in \mathcal{U}$, we call $|u|$ the *height* (or *generation*) of u. For $u = (u_1, \dots, u_k) \in \mathcal{U} \backslash \{\varnothing\}$, let the *parent of* u be $p(u) := (u_1, \dots, u_{k-1})$.

An *ordered, rooted tree* $\mathcal{T}$ is a subset of $\mathcal{U}$, such that $\varnothing \in \mathcal{T}$, and for any $u = (u_1, \ldots, u_k) \in \mathcal{T}$ we have $p(u) \in \mathcal{T}$ and $(u_1, \ldots, u_{k-1}, j) \in \mathcal{T}$ for $1 \leq j \leq u_k - 1$. For each $u \in \mathcal{T}$, let the *children* of u in $\mathcal{T}$ be $C(u) := \{w \in \mathcal{T} : p(w) = u\}$, and set $c(u) := |C(u)|$. When we add edges between each $u \in \mathcal{T}$ and all of its children $C(u)$, the resulting graph is indeed acyclic and connected.

Let μ be a probability distribution on $\mathbb{N}_0$. A *Galton–Watson tree* (or a *branching process tree*) $\mathcal{T}_\mu$ with offspring distribution μ is a random ordered, rooted tree generated as follows. First, sample independent random variables $\bar{c}(u)$ distributed as μ, for every $u \in \mathcal{U}$. Then let $\mathcal{T}_\mu$ be the unique ordered, rooted tree in which the number of children $c(u)$ in $\mathcal{T}_\mu$ is equal to $\bar{c}(u)$ for every $u \in \mathcal{T}_\mu$.

We will refer to $|\mathcal{T}_\mu|$ as the *total population size* of $\mathcal{T}_\mu$, in keeping with Galton's biological motivation for the model. We define *extinction* to be the event $\{|\mathcal{T}_\mu| < \infty\}$, and *survival* to be the event $\{|\mathcal{T}_\mu| = \infty\}$.

Remark. The classical definition of the *Galton–Watson process* is a Markov chain $(Z_n)_{n\geq 0}$ on $\mathbb{Z}_{\geq 0}$ characterised by $Z_0 = 1$, and the recurrence $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}$, where $(\xi_i^{(n)}, n \geq 0, i \geq 1)$ are IID random variables with distribution μ. Each Z_n represents the number of individuals in the nth generation. Note that this definition can be recovered from the Galton–Watson tree $\mathcal{T}_\mu$ by taking $Z_n = |\{u \in \mathcal{T}_\mu : |u| = n\}|$.

The fundamental result concerning survival of Galton–Watson processes says that positivity of the survival probability depends only on the mean of μ. First, we define $f_\mu(t) = \sum_{k=0}^{\infty} \mu_k t^k$ to be the probability generating function corresponding to μ. Now we can state the following result.

Proposition 1.6. Let μ be a probability distribution on $\mathbb{Z}_{\geq 0}$ other than δ_1. Then, if $\sum_{k\geq 0} k\mu_k \leq 1$,

$$\mathbb{P}(|\mathcal{T}_\mu| = \infty) = 0.$$

If $\sum_{k\geq 0} k\mu_k > 1$, then there is a unique $t_\mu \in [0, 1)$ such that $f_\mu(t_\mu) = t_\mu$. Furthermore

$$\mathbb{P}(|\mathcal{T}_\mu| = \infty) = \zeta_\mu := 1 - t_\mu.$$

Definition 1.7. We say a Galton–Watson process is *subcritical*, *critical* or *supercritical* when $\sum_{k\geq 0} k\mu_k$ is, respectively, less than 1, equal to 1, or greater than 1.

In the case where μ is Poisson(c), the pgf f_μ is particularly tractable, so let us write ζ_c for the survival probability in this case. We obtain

$$\zeta_c = 1 - e^{-c\zeta_c}. \tag{1.2}$$

Later, we will be interested in graphs in the *barely supercritical* regime where $c = 1+\epsilon$ for $0 < \epsilon \ll 1$. Letting $\epsilon \downarrow 0$ and linearising in (1.2), we obtain $\zeta_{1+\epsilon} = 2\epsilon(1+o(1))$.

1.1.3 Uniform trees

Recall that a *tree* is a simple connected graph with no cycles. Cayley's formula states for that every N, there are N^{N-2} trees on vertex set $[N]$. We will call T_N, a tree chosen uniformly at random from this set, a *uniform tree on* $[N]$. This is a special case of a *uniform spanning tree*, here with reference to K_N, the complete graph on $[N]$.

Now consider $G(N,p)$, and fix $k \leq N$. Conditional on the event that $\{1,\dots,k\}$ is the vertex set of a tree in $G(N,p)$, this tree is a uniform tree on $\{1,\dots,k\}$, since, in $G(N,p)$, the probability of any configuration depends only on the number of edges present.

Some of the tools required to treat uniform trees will be easier to develop in the setting of Galton–Watson trees. Fortunately, it is possible to describe a uniform tree as a Galton–Watson tree conditioned to have a particular total population size. We formalise this, and give a short proof, based on Aldous's outline [3].

Proposition 1.8. [3, §2.1] Let μ be Poisson(1), and $\mathcal{T}_\mu$ the corresponding Galton–Watson tree. We condition on the event that $|\mathcal{T}_\mu| = N$, and then assign labels $[N]$ to the vertices of $\mathcal{T}_\mu$ uniformly at random. To the resulting (labelled) ordered, rooted tree, we associate the corresponding *unordered*, labelled tree with edges precisely from vertex u to each of its $c(u)$ children, for all $u \in \mathcal{T}_\mu$. We call this unordered, labelled tree $\tilde{T}_N$. Then $\tilde{T}_N \overset{d}{=} T_N$.

Proof. For any rooted, ordered tree $\mathcal{T}$ with size N, we have

$$\mathbb{P}(\mathcal{T}_\mu = \mathcal{T}) = \prod_{v\in\mathcal{T}} \frac{e^{-1}}{C(v)!} = e^{-N} \prod_{v\in\mathcal{T}} \frac{1}{C(v)!}.$$

Since $\mathbb{P}(|\mathcal{T}_\mu| = N)$ is a function of N,

$$\mathbb{P}(\mathcal{T}_\mu = \mathcal{T} \,|\, |\mathcal{T}_\mu| = N) = f(N) \prod_{v\in\mathcal{T}} \frac{1}{C(v)!},$$

where $f(N)$ depends only on N, not on $\mathcal{T}$.

Now consider a particular *unordered*, labelled rooted tree $(\tilde{T}, \tilde{\rho})$ with $|\tilde{T}| = N$ and a root $\tilde{\rho} \in \tilde{T}$. Then there are $\prod_{v\in\tilde{T}} C(v)!$ ordered, labelled rooted trees associated to $(\tilde{T}, \tilde{\rho})$. Each of these ordered labelled rooted trees rooted trees arises with probability $\frac{f(N)}{N!} \prod_{v\in\mathcal{T}} \frac{1}{C(v)!}$ when we choose a uniformly labelled version of $\mathcal{T}_\mu$ conditioned to have N vertices.

Therefore $\mathbb{P}\Big((\tilde{T}_N, \tilde{\rho}_N) = (\tilde{T}, \tilde{\rho})\Big) = \frac{f(N)}{N!}$. This is a function of N alone, so the law of $(\tilde{T}_N, \tilde{\rho}_N)$ is uniform on the set of rooted unordered labelled trees, as required. □

1.1.4 Multiplicative coalescence

When we study the random graph process as above, sometimes we will be interested only in the sizes of components, rather than the graph structure within such components. Ignoring this internal structure, what remains is a process of block sizes, where pairs of blocks can join together, corresponding to the addition of an edge between two hitherto distinct components. Various models of such a *coalescence process* have been studied mathematically, and are applicable here.

Continuum models and Smoluchowski's equations

Consider a general setting with a large number of blocks, which may have any mass $x \in \mathbb{R}_+$. Any pair of blocks with masses x and y may merge to form a single block of mass $x + y$, and the rate at which this happens is specified by some *kernel* $K(x, y) \geq 0$.

Now, we assume that all masses are positive integers. Let $c_k(t) \geq 0$ be a function which, heuristically, represents the density of blocks with mass $k \in \mathbb{N}$ at time t. We may describe the evolution through a system of coupled ODEs, referred to as *Smoluchowski's (coagulation) equations*:

$$\frac{\mathrm{d}}{\mathrm{d}t}c_k(t) = \frac{1}{2}\sum_{\ell=1}^{k-1} K(\ell, k-\ell)c_\ell(t)c_{k-\ell}(t) - c_k(t)\sum_{\ell=1}^{\infty} K(k,\ell)c_\ell(t), \quad k \geq 1. \tag{1.3}$$

In each of these equations, the first term on the RHS gives the rate at which blocks of mass k are formed as a result of two smaller blocks merging, and the second term gives the rate at which blocks of mass k are lost as a result of merging with other blocks.

Smoluchowski [63] introduced this model to investigate molecules in solution, for which a typical kernel might be $K(x,y) := (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$. However, three simpler kernels with combinatorial interpretations have proven tractable and useful for applications. The constant kernel $K(x,y) \equiv 1$ leads to *Kingman's coalescent* [39], which has been much studied as a model in population genetics. The *additive* coalescent with kernel $K(x,y) = x + y$, studied by Aldous and Pitman [6], and Evans and Pitman [25], is related to a natural growth process for uniform forests [58]. These and many other examples of coalescence are considered from both mathematical and applied perspectives in Aldous's survey paper [7].

For the purposes of this thesis, we focus on the *multiplicative kernel* $K(x,y) = xy$. The relevance of this kernel is based on the following observation. In the random graph process, whenever there are distinct components with sizes x and y, the rate at which they are joined is proportional to the number of potential edges between them, that is xy.

In this setting, instead of $c_k(t)$, it is more convenient to consider $v_k(t) \geq 0$, the density of mass contained in blocks of mass k at time t. So $v_k(t) = kc_k(t)$, and we rewrite (1.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}v_k(t) = \frac{k}{2}\sum_{\ell=1}^{k-1} v_\ell(t)v_{k-\ell}(t) - kv_k(t)\sum_{\ell=1}^{\infty} v_\ell(t), \quad k \geq 1. \tag{1.4}$$

Some arguments and interpretations are better suited to one of $(c_k(t))$ and $(v_k(t))$ than the other. For example, in Chapter 4, it will be convenient to state the main result about convergence of the mean-field frozen percolation process in terms of $(v_k(t))$, but for technical reasons the proof will use $(c_k(t))$.

Solutions to (1.4) and the gelation property

There are analytic challenges in proving existence and uniqueness of solutions to (1.4) for various classes of initial condition $(v_k(0), k \in \mathbb{N})$. McLeod [50] demonstrated existence and uniqueness of solutions on $t \in [0, 1)$ under *monodisperse initial conditions*, that is when $v(0) = (1, 0, 0, \dots)$. This corresponds to starting from an empty graph. The proof relies on the finiteness of $\sum_{k=1}^{\infty} kv_k(t)$, which diverges as $t \uparrow 1$, corresponding to the formation of a giant component in the random graph process at criticality.

This *gelation* effect holds more generally for solutions to (1.4). That is, there exists some *gelation time* $T_g \geq 0$, depending on the initial condition $v(0)$, such that the total mass $\sum_{k=1}^{\infty} v_k(t)$ is constant for $t \in [0, T_g]$, but strictly decreasing on $[T_g, \infty)$. It is helpful to think of this loss of mass arising from the creation (in finite time) of blocks 'with infinite mass', which can't participate in further coalescence events, and don't contribute to the sum $\sum_{k=1}^{\infty} v_k(t)$.

Kokholm [40] was able to extend existence and uniqueness of the solution to (1.4) to the entirety of $t \in \mathbb{R}_{\geq 0}$, again with monodisperse initial conditions. This complex-analytic argument exploits the exact form of the solution with these initial conditions, and does not generalise easily to a broader class of initial conditions.

With general initial conditions, existence and uniqueness of solutions to (1.4) up to the gelation time was shown by several authors, including Norris [54] under more general coalescence kernels. Existence of solutions beyond the gelation time was studied by several authors, including Laurençot [41] via PDE methods, and Jeon [35] via a weak convergence argument similar to the one we will use in Chapter 4. All of these authors show the following expression for the gelation time

$$T_g = \frac{1}{\sum_{k\geq 1} k v_k(0)} \in [0, \infty). \tag{1.5}$$

Flory's model [26] is an alternative to Smoluchowski's, in which the 'infinite-mass' *gel* continues to coalesce with finite blocks (called the *sol*). The corresponding equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} v_k(t) = \frac{k}{2} \sum_{\ell=1}^{k-1} v_\ell(t) v_{k-\ell}(t) - k v_k(t) \sum_{\ell=1}^{\infty} v_\ell(0). \tag{1.6}$$

Here, the final term $\sum_{\ell=1}^{\infty} v_l(0)$ represents the 'total mass' of the system, which is assumed to be constant. Note that this model corresponds directly to the random graph process, where the giant component continues to form edges with small components. Norris [55] shows existence and uniqueness of solutions to a version of (1.6) corresponding to a substantial generalisation of the coalescence dynamics, called *cluster coagulation*.

Global existence and uniqueness of solutions to (1.4) under general initial conditions was unresolved until Ráth [60] proved this with the restriction that the initial $v(0)$ has finite support. In a similar argument via generating functions, Normand and Zambotti [53] show the same result without the requirement that $v(0)$ has finite support. They further show that the total mass $\Phi(t) := \sum v_k(t)$ is analytic on $[0, \infty)\backslash\{T_g\}$, and give conditions under which its right-derivative at criticality satisfies $\frac{\mathrm{d}_+}{\mathrm{d}t}\Phi(T_g) < \infty$. We present a version of this result in Chapter 4.

Stochastic coalescents

We are primarily interested in solutions to Smoluchowski's equations as limits of random, discrete coalescent processes. Such processes were introduced by Marcus [49] and studied by Lushnikov [46, 47], and are often called *Marcus–Lushnikov processes*.

We let $N \in \mathbb{N}$ be some index, which we treat as the scaling for the total mass. Then, we consider the Markov chain with state space given by finite non-increasing sequences of positive integers (which we interpret as block masses) and the following transitions: at rate $K(x, y)/N$, we replace each pair x and y in the sequence with $x + y$, reordering if necessary. The sequence of component sizes in a random graph process (including a version started from any initial graph on $[N]$) is an example of such a Marcus–Lushnikov process with multiplicative kernel.

Given such a process, let $c_k^N(t) := \frac{1}{N}\#\{\text{blocks with mass } k \text{ at time } t\}$, and $v_k^N(t) := kc_k^N(t)$. In many cases, it is natural to assume that N is the initial total mass, that is $\sum_{k=1}^{\infty} v_k^N(0) = 1$. This is consistent with the interpretation of the model as N particles of equal mass, some of which join together into blocks. Note then that $c^N(t)$ can be viewed as a measure on block masses, and $v^N(t)$ the corresponding size-biased measure (or probability distribution) giving the mass of the block containing a randomly-selected particle.

We are interested in the convergence of these Marcus–Lushnikov processes as the index N tends to infinity. Jeon [35] and Norris [54] show convergence to the Smoluchowski equations (1.3) for a general class of kernels which do not induce gelation. In particular, this does not include the multiplicative case. Norris [55] shows as a special case of subsequent work on *cluster coagulation* that multiplicative Marcus–Lushnikov processes converge uniformly in distribution in ℓ_1 to the solution of Flory's equations (1.6), and several authors [27, 28] have studied in greater detail which kernels lead to Flory rather than Smoluchowski equations as the hydrodynamic limit.

In Section 1.3.1, we will introduce a mean-field frozen percolation model, which shares some of the dynamics of the multiplicative coalescent but which *does* have Smoluchowski equations as its hydrodynamic limit.

1.2 Exploration processes and limits

1.2.1 Exploration processes

Definition 1.9. Let G be some (not necessarily random) graph with vertex set $[N]$. Throughout, we define the *neighbourhood* of a vertex v to be $\Gamma(v) := \{w \in V(G), wv \in E(G)\}$, the set of vertices connected to v by an edge. Then, take a (possibly random) ordering $v_1, v_2, \ldots, v_N$ of the vertices such that for every $m = 0, 1, \ldots, N-1$,

$$v_{m+1} \in \Big(\Gamma(v_1) \cup \ldots \cup \Gamma(v_m)\Big) \backslash \{v_1, \ldots, v_m\}, \tag{1.7}$$

whenever this set is non-empty. We now define the *exploration process* as

$$\begin{cases} S_0 & = \ 1 \\ S_{m+1} & = \ S_m + \Big|\Gamma(v_{m+1}) \backslash (\{v_1\} \cup \Gamma(v_1) \cup \ldots \cup \Gamma(v_m))\Big| - 1, \quad m \geq 0. \end{cases} \tag{1.8}$$

We may define

$$H_k := \inf\{m : S_m = -k+1\}, \tag{1.9}$$

and then $(H_1, H_2 - H_1, H_3 - H_2, \ldots)$ is the sequence of component sizes in G, in some order. In particular, for $1 \leq m \leq H_1$, it can be seen from (1.8) by induction that $S_m = |\mathcal{Z}_m|$, where

$$\mathcal{Z}_0 := \{v_1\}, \quad \mathcal{Z}_m := \Big(\Gamma(v_1) \cup \ldots \cup \Gamma(v_m)\Big) \backslash \{v_1, \ldots, v_m\}, \quad m \geq 1. \tag{1.10}$$

Heuristically, we imagine 'exploring' the graph one vertex $v_1, v_2, \ldots$ at a time, revealing neighbours as we go. Then, when we are at vertex v_m, $\mathcal{Z}_m$ is the set of vertices we have *seen but not visited*, which we will sometimes refer to as the *stack*. In Chapter 3 it will be more convenient to consider the *reflected exploration process* $(Z_m)_{m \geq 0}$ defined by $Z_m = |\mathcal{Z}_m|$, for $m \geq 0$. By construction, this is non-negative, and satisfies

$$Z_m = 1 + S_m - \min_{k \leq m} S_k, \quad m \geq 0. \tag{1.11}$$

Examples of orderings

- Aldous [5] uses a *breadth-first ordering* to investigate the distribution of component sizes in critical random graphs. We define such an ordering as follows.

  We assume the graph has vertex set $[N]$. Let v_1 be chosen uniformly at random from $[N]$. If v_1 has exactly k neighbours, we let $v_2, \ldots, v_{k+1}$ be these neighbours in increasing order. Then if v_2 has exactly l neighbours different to $v_1, \dots, v_{k+1}$, we let $v_{k+2}, \dots, v_{k+l+1}$ be these neighbours in increasing order. We continue this procedure. At any time when $\{v_1, \dots, v_m\}$ is a union of connected components of the graph, we choose v_{m+1} uniformly at random from the remaining vertices, and continue until all vertices have been chosen.

In Chapter 2, we use such a breadth-first ordering, but where $v_2, \dots, v_{k+1}$ are chosen in uniformly random order, and similarly for each set of new neighbours. This simplifies some proofs by making the exploration process exchangeable.

- A similar construction can be used for a *depth-first ordering*, where we consider the descendents of the first offspring of a vertex before we consider any other offspring or their descendents. Le Gall [42] uses such an ordering $v_1, v_2, \dots, v_N$ for a tree T_N of size N, and considers the *height process*

$$h_0^N := 0, \quad h_m^N := d_{T_N}(v_1, v_m), \quad m = 1, \dots, N, \tag{1.12}$$

  where d_{T_N} is the usual graph distance on T_N. In many circumstances, we can profitably extract the height process from the depth-first exploration process. We will discuss shortly some of the results about limits of trees which follow from these ideas.

- A further option is to choose an ordering uniformly at random. We can do this iteratively. Choose v_1 uniformly from $V(G)$. Then for each $m \geq 1$ in turn, we choose v_{m+1} uniformly from $\mathcal{Z}_m$ as in (1.10), unless $\mathcal{Z}_m = \varnothing$, in which case we choose v_{m+1} uniformly from $V(G) \backslash \{v_1, \dots, v_m\}$. We will use this ordering in Chapter 3 for a *multitype exploration process* for a graph where each vertex has a type in $[k]$. Here the exploration process is $\mathbb{Z}^k$-valued. The calculations in this chapter remain valid under alternative orderings, but this ordering ensures the exploration process is Markov.

In these examples, whenever we exhaust a component, we choose the next vertex uniformly at random. There are other possibilities: for example, one could choose the vertex with the smallest label that hasn't yet been considered. However, an advantage of the uniform choice is that $(H_1, H_2 - H_1, H_3 - H_2, \ldots)$ is the sequence of component sizes of G in *size-biased order*.

1.2.2 Exploring random graphs and trees

The heuristic for exploring a random graph is to 'reveal' the randomness one vertex at a time. The exploration process will be particularly tractable when $(S_m)_{m\geq 0}$ is Markov.

Galton–Watson trees

Let $\mathcal{T}_\mu$ be the Galton–Watson tree with offspring distribution μ, as introduced previously, and let $v_1, v_2, \ldots$ be any ordering of its vertices satisfying $v_1 = \varnothing \in \mathcal{U}$ and (1.7), as well as the following condition. We insist that, conditional on $(v_1, \ldots, v_m)$ and $\mathcal{Z}_m$,

$$\text{the choice of } v_{m+1} \text{ is independent of } \mathcal{T}_\mu \text{ restricted to } \mathcal{U}\backslash\{v_1, \ldots, v_m\}. \tag{1.13}$$

So the choice of v_{m+1} may depend on external randomness independent of the tree. However, this non-look-forward condition says, informally, that the choice of v_{m+1} depends only on the structure of the subset of the tree which has already been explored, and not on the descendents of the current stack $\mathcal{Z}_m$.

Let $(S_m)_{m\geq 0}$ be the corresponding exploration process, which we can rewrite as

$$\begin{cases} S_0 &= 1 \\ S_{m+1} &= S_m + c(v_{m+1}) - 1, \quad m \geq 0. \end{cases} \tag{1.14}$$

The number of children $c(v_{m+1})$ is independent of the number of children of previously-explored vertices, and so we conclude that $(S_m)_{m\geq 0}$ is Markov and has IID increments distributed as $\mu - 1$.

Remark. The condition (1.13) includes all the examples of orderings given in Section 1.2.1. However, it does not include, for example, the situation where, conditional on $(v_1, \ldots, v_m)$ and $\mathcal{Z}_m$, z_{m+1} is chosen to be a vertex $v \in \mathcal{Z}_m$ for which $c(v)$ is maximal.

In general, the exploration process corresponding to this ordering does not have the Markov property nor IID increments.

In particular, if we take

$$S_m = 1 - m + \sum_{i=1}^{m} \xi_i, \quad m \geq 0,$$

where $\xi_1, \xi_2, \ldots$ are IID with distribution μ, this has the same distribution as the original exploration process, and thus gives us a method to establish the distribution of the total population size of $\mathcal{T}_\mu$, via (1.9). We use the following result of Dwass [20].

Proposition 1.10 (Hitting time theorem)**.** Let $(S_m)_{m\geq 0}$ be a random walk starting at $k \geq 1$, with IID increments supported on $\{-1, 0, 1, 2, \ldots\}$. Then, with $H_1 := \min\{m : S_m = 0\}$ as before,

$$\mathbb{P}(H_1 = N) = \frac{k}{N}\mathbb{P}(S_N = 0). \tag{1.15}$$

Taking this result in the case $k = 1$, we obtain

$$\mathbb{P}(|\mathcal{T}_\mu| = N) = \mathbb{P}(H_1 = N) = \frac{1}{N}\mathbb{P}(\xi_1 + \ldots + \xi_N = N - 1).$$

Sketch proof of Proposition 1.10. We outline a proof for the case $k = 1$. See [69] for an argument for general $k \geq 1$ along similar lines. Let $x_1, \ldots, x_N$ be a sequence of integers such that each $x_i \geq -1$ and $x_1 + \ldots + x_N = -1$. Then, it is straightforward to check that there is exactly one cyclic reordering $x'_1, \ldots, x'_N$ of $x_1, \ldots, x_N$ such that

$$x'_1 + \ldots + x'_m \geq 0, \quad m = 1, \ldots, N - 1.$$

Indeed, let m be the first index for which $x_1 + \ldots + x_m$ achieves its minimum. Then $x'_i := x_{m+i}$ (with indices taken modulo N) is the unique cyclic reordering with this property.

Since IID increments $\xi_1, \ldots, \xi_N$ are exchangeable, (1.15) follows. □

Erdős–Rényi random graph

We may apply a similar approach to $G(N,p)$. We consider orderings of $[N]$ satisfying (1.7) and a non-look-forward condition similar to (1.13). That is, conditional on $(v_1,\dots,v_m)$ and $\mathcal{Z}_m$,

$$\text{the choice of } v_{m+1} \text{ is independent of } G(N,p) \text{ restricted to } [N]\backslash\{v_1,\dots,v_m\}. \quad (1.16)$$

Then, given $v_1,\dots,v_{m+1}$ and $\Gamma(v_1)\cup\ldots\cup\Gamma(v_m)$, we know that $\Gamma(v_{m+1})\backslash(\{v_1\}\cup\Gamma(v_1)\cup\ldots\cup\Gamma(v_m))$ includes each vertex in

$$[N]\backslash(\{v_1\}\cup\Gamma(v_1)\cup\ldots\cup\Gamma(v_m)\cup\{v_{m+1}\}) = [N]\backslash(\{v_1,\dots,v_{m+1}\}\cup\mathcal{Z}_m)$$

independently with probability p. Therefore, conditional on $(Z_0,Z_1,\dots,Z_m)$, we have from (1.8)

$$S_{m+1} - S_m + 1 \overset{d}{=} \mathrm{Bin}(N-m-(Z_m\vee 1),p). \quad (1.17)$$

Note that the term $Z_m \vee 1$ appears because $v_{m+1}\in\mathcal{Z}_m$ iff $Z_m\geq 1$. In particular, for all m, and conditional on any sequence of values for $(Z_0,Z_1,\dots,Z_m)$, we have

$$S_{m+1} - S_m + 1 \leq_{st} \mathrm{Bin}(N-1,p).$$

Observe now that the distributions $\mathrm{Bern}(1-e^{-t/N})$ and $\mathrm{Po}(t/N)$ place the same probability mass on zero, and so $\mathrm{Bern}(1-e^{-t/N}) \leq_{st} \mathrm{Po}(t/N)$. So, in the particularly relevant case $p=1-e^{-t/N}$, we have, again conditional on any sequence $(Z_0,Z_1,\dots,Z_m)$,

$$S_{m+1} - S_m + 1 \leq_{st} \mathrm{Po}\Big(\tfrac{N-1}{N}t\Big) \leq_{st} \mathrm{Po}(t). \quad (1.18)$$

We can use these estimates to bound in probability the size of the first component seen in the exploration process, and thus the largest component in $G(N,p)$. We will use such an argument for a particular class of inhomogeneous random graphs in Chapter 3.

In the case of $G(N,p)$, it is not possible to recover the exact graph structure from the exploration process, since not all (potential) edges are considered during the process. In particular, for each m, it is possible that there are edges between v_{m+1} and $\mathcal{Z}_m \setminus \{v_{m+1}\}$. Any such edge forms a cycle with pre-existing edges, and is present independently with probability p. So, the distribution of $G(N, p)$ can be recovered from the distribution of the exploration process, by adding each of these potential cyclic edges independently with probability p.

The seminal example of exploring $G(N, p)$ is Aldous's study [5] of the *critical window*, where $p = (1 + \lambda N^{-1/3})/N$. We will introduce this approach fully in Chapter 2, where we extend the argument to a forest-valued version of the same random graphs.

1.2.3 Local limits

For large N, and small values of m, the stochastic bound given in (1.18) is a good approximation to the distribution of each increment in the exploration process. In other words, initially the exploration process of $G(N, 1 - e^{-t/N})$ is very close in distribution to the exploration process of the Galton–Watson tree with Poisson(t) offspring distribution. So the local graph structure of $G(N, 1 - e^{-t/N})$ near some uniformly-chosen vertex is similar in distribution (for large N) to the Poisson Galton–Watson tree. The following definition, introduced by Benjamini and Schramm [12], makes this notion precise.

First, we say that two rooted graphs (G, ρ) and (G', ρ') are *isomorphic* if there is a graph isomorphism from G to G' that maps ρ to ρ'. Given a rooted graph (G, ρ), and an integer $R \geq 0$, we denote by $B_R(G, \rho)$ the induced rooted subgraph of G consisting of all vertices $v \in G$ with $d_G(\rho, v) \leq R$, still rooted at ρ.

Definition 1.11. Consider a sequence (G_n, ρ_n) of random finite rooted graphs. We say that a random rooted graph (G, ρ) is the *local weak limit* of (G_n, ρ_n) if for all finite rooted graphs (H, ρ_H), and all finite R, the probability that $B_R(G_n, \rho_n)$ is isomorphic to (H, ρ_H) converges to the probability that $B_R(G, \rho)$ is isomorphic to (H, ρ_H) as $n \to \infty$.

Consider now a sequence (G_n) of random finite unrooted graphs, and let ρ_n be a uniformly-chosen root in G_n. We say a random locally-finite rooted graph (G, ρ) is the *Benjamini–Schramm limit* of (G_n) if it is the local weak limit of (G_n, ρ_n).

Examples of weak local limits

- Following Benjamini and Schramm [12], we illustrate why the choice of root is important. Let B_n be the binary tree with n layers, that is, for which $|V(B_n)| = 2^n - 1$. When the usual root is chosen, the local weak limit is the infinite binary tree. When the root ρ_n is chosen uniformly at random, the distance of ρ_n from the layer of leaves in B_n converges to a geometric distribution with parameter 1/2. So the Benjamini–Schramm limit of B_n is not the infinite binary tree, but instead the canopy tree, with a particular root distribution.

  We define the *canopy tree* by construction. Start with the graph on $\mathbb{Z}_{\geq 0}$ where there is an edge between m and n precisely when $|m-n| = 1$. Then, for each $n \geq 1$, we add an edge between vertex n and the root of a copy of B_n. The resulting graph is the canopy tree. Note that if we delete the edge between vertices n and $n+1$, this splits the tree into two components where the finite component is isomorphic to B_{n+1}. To complete the description of the Benjamini–Schramm limit of B_n, we specify the distribution of the root of the canopy tree by $\mathbb{P}(\rho = n) = \frac{1}{2^{n+1}}$, for every vertex $n \in \mathbb{Z}_{\geq 0}$.

- Consider the Galton–Watson tree $\mathcal{T}_\mu$ corresponding to a critical offspring distribution μ with $\sum k\mu_k = 1$ and finite variance. Then for each $n \geq 1$, let T_n be the random rooted tree given by conditioning $\mathcal{T}_\mu$ to have total population size equal to n. Then the local weak limit as $n \to \infty$ is the *size-biased Galton–Watson tree*, an infinite tree introduced by Kesten [38], where vertices have one of two types: vertices on the (unique) infinite *spine* have the size-biased version of μ as their offspring distribution; and other vertices have the usual offspring distribution.

- The Benjamini–Schramm limit of $G(N, c/N)$ is the Galton–Watson tree with Poisson(c) offspring distribution. This comparison is particularly useful in the supercritical regime, where $c > 1$, for which, with high probability, the size of the largest component $|\mathcal{C}_1^N| = (1 + o(1))\zeta_c N$ with high probability, where ζ_c is the survival probability of the corresponding Galton–Watson process. The following heuristic applies. From the local limit, ζ_c gives the probability that a uniformly-chosen vertex is in a 'large' component, and then asymptotically almost all vertices in such large components are, with high probability, in the same giant component. Full details of this argument can be found in Section 4.4 of van der Hofstad's book [68].

1.2.4 Scaling limits

As well as the typical local structure of large trees and graphs, we are also interested in the global properties of such objects. In some circumstances one can characterise asymptotic properties of trees via some continuous 'tree-like' limit object, such as Aldous's Brownian continuum random tree [3].

There are several ways to characterise this convergence formally. One might show that some functionals of the large trees, such as the exploration processes we have introduced, converge in distribution as processes (possibly after rescaling) to some corresponding process associated with the limit tree. For example, consider the height process $(h^N_m,\ m = 0, 1, \ldots, N)$ for a uniform rooted tree on $[N]$, as defined at (1.12). We use the formulation of Le Gall [42], although Aldous [4] shows the same result for the closely-related *contour process.*

THEOREM 1.12. [42, Theorem 1.15] Let $(B^{\mathrm{ex}}(t), 0 \le t \le 1)$ be a standard Brownian excursion. Then

$$\frac{1}{\sqrt{N}}\Big(h^N_{\lfloor Nt\rfloor}, 0 \le t \le 1\Big) \quad \stackrel{d}{\to} \quad 2(B^{\mathrm{ex}}(t), 0 \le t \le 1), \tag{1.19}$$

as $N \to \infty$, in $\mathbb{D}([0,1], \mathbb{R}_{\ge 0})$, the space of non-negative càdlàg functions with the Skorohod topology.

More recently, there has been much interest in showing convergence of the trees themselves, viewed as metric spaces. Le Gall [43] shows a version of (1.19) for the trees themselves with respect to the Gromov–Hausdorff topology. Many subsequent authors have extended these results to other settings, including Addario–Berry, Broutin and

Goldschmidt [1], who show a metric space limit for the connected components of $G(N,p)$ in the critical window $p = (1 + \lambda N^{-1/3})/N$.

For the purposes of this thesis, convergence of exploration processes is all we require. In particular, consider the exploration process $(S_m^N,\ m = 0, 1, \ldots, N)$ derived from the breadth-first ordering of a uniform tree on $[N]$ where in each round the new stack vertices are ordered uniformly at random, as in the first example of Section 1.2.1. Then we have a similar result

Proposition 1.13. As $N \to \infty$

$$\frac{1}{\sqrt{N}}\left(S^N_{\lfloor Nt \rfloor},\ 0 \le t \le 1\right) \quad \overset{d}{\to} \quad 2(B^{\text{ex}}(t),\ 0 \le t \le 1), \tag{1.20}$$

in $\mathbb{D}([0,1], \mathbb{R}_{\ge 0})$.

By Proposition 1.8, it suffices to check the corresponding result for Galton–Watson trees conditioned to have size N. But the exploration process for a Galton–Watson tree has IID increments as in (1.14). Kaigh [37] proves a version of Donsker's theorem for convergence of random walks conditioned on a particular large value for the hitting time of zero, and we can use this to derive the result (1.20) directly.

Note. Though we state Proposition 1.13 for a specific ordering, it will hold for the exploration processes corresponding to any ordering satisfying a non-look-forward condition analogous to (1.13) and (1.16). This version is all we require in Chapter 2, and avoids the requirement to introduce notation to explain in general how to pass between the non-look-forward condition in the uniform tree and the non-look-forward condition in the Galton–Watson tree.

1.3 Self-organised criticality

In nature, we observe systems which are updated by simple local rules, where occasionally the effects of a single event spread quickly through a large proportion of the system. Bak, Tang and Wiesenfeld [11] introduced the *sandpile model* as an intuitively simple mathematical example. Here, piles of sand accumulate at each point on some lattice, until the height of a pile exceeds some threshold value. At this point the pile collapses completely, and all its grains of sand are redistributed equally among neighbouring piles, which may then themselves collapse. In this way, a single extra grain may trigger an 'avalanche' which spreads through the entire system. This property is sometimes termed *long-range correlation*, as parts of the system are occasionally affected by events which start a large distance away.

These same authors coined the phrase *self-organised criticality* in [10] to describe the qualitative properties of this model, where the system is driven to criticality by the dynamics, from a broad class of initial conditions. Criticality could be characterised in several ways, but the easiest condition to verify across a broad range of models is that the distribution of the size of an event in the model has a power-law tail. In the case of sandpiles, this is taken to be the number of sites affected by an avalanche. In a graph-based model, it could be the sizes of components, or the sizes of components affected by a destructive event.

Note that the distribution of component-sizes in $G(N, 1/N)$ has a power-law tail. However, in the Erdős–Rényi random graph, this criticality is achieved only by fine-tuning the parameter, rather than through self-organisation. In the main examples, the local dynamics are mostly monotonic, with occasional macroscopic destructive events acting in the opposite direction to prevent the system becoming supercritical.

Our main focus will be on mean-field frozen percolation and forest fires, two versions of the random graph process which exhibit self-organised criticality.

1.3.1 Mean-field frozen percolation

The *frozen percolation process* on a graph G was introduced by Aldous [8] in the case where G is the infinite binary tree. Informally, we perform classical percolation, with the restriction that 'components are not allowed to participate in the dynamics once they become infinite'. It is important to note that when G is an infinite graph, constructing a well-defined version of such a process is technically challenging, and sometimes impossible.

A *mean-field* version of frozen percolation on the complete graph on N vertices was considered by Ráth [60] as a modification of the Erdős–Rényi random graph process. Initially, all the N vertices are declared to be *alive*, and there may be some (possibly randomly chosen) edges present. Between any pair of alive vertices, edges appear at rate $1/N$, as in the random graph process, but each vertex also carries an independent exponential clock with rate $\lambda(N)$. When the clock rings, we say the vertex has been *struck by lightning*, and this vertex, and all vertices in its component at that time are declared *frozen* (or *dead*). Frozen vertices can never become alive again, so as time advances, the number of alive vertices decreases. Throughout this thesis, we will assume that the lightning rate is chosen with *critical scaling* such that

$$1/N \ll \lambda(N) \ll 1. \tag{1.21}$$

The heuristic is that for large N, small components are never struck by lightning, while giant components are immediately struck by lightning.

We let $\mathcal{G}^N(t)$ be the graph of alive vertices at time t, and

$$v_k^N(t) := \frac{1}{N}\#\{\text{alive vertices in size } k \text{ components at time } t\}, \quad k \geq 1,\, t \geq 0.$$

We note that $(v^N(t), t \geq 0)$ is a Markov process on ℓ_1. Typically we consider such a process for $N \in A$, some infinite subset of $\mathbb{N}$, and limiting behaviour as $N \to \infty$, when $v^N(0)$ converges to a limiting initial distribution $v(0)$.

Ráth [60] shows that when $v^N(0) = v(0)$ for all $N \in A$, and $v(0)$ has finite support, then $v^N \xrightarrow{d} v$ as $N \to \infty$ in $\mathbb{D}([0,T], \ell_1)$ for any $T > 0$. Here $v = (v_k(t),\, k \geq 1,\, t \geq 0)$ is the (unique) solution to Smoluchowski's equation (1.4). The self-organised criticality of this model is described via the tails of the limit process, that is the solution to (1.4). Ráth shows that for any solution to (1.4) where the initial distribution $v(0)$ has finite support, for any $t \geq T_g$,

$$\sum_{\ell \geq k} v_k(t) = \Theta\Big(k^{-1/2}\Big), \quad k \to \infty. \qquad \text{([60], Theorem 1.5)}$$

We note the requirement that this deterministic initial distribution $v^N(0) = v(0)$ apply for all N. Since $v^N(0) \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}/N$ by construction, except in the monodisperse case $v(0) = (1, 0, 0, \ldots)$, this forces us to consider only $\mathbb{Q}_{\geq 0}^{\mathbb{N}}$-valued $v(0)$, and convergence only along an appropriate subsequence.

In Chapter 4, we will introduce this result in more detail and prove a version for a sequence of mean-field frozen percolation models with initial conditions suitable for our application, in particular without the requirement that $v(0)$ have finite support.

1.3.2 Mean-field forest fires

The *mean-field forest fire* is a similar process on the complete graph on N vertices, introduced by Ráth and Tóth [61], defined as follows. As in mean-field frozen percolation, the initial graph on $[N]$ may include some edges (possibly randomly chosen). Between any pair of vertices, edges appear at rate $1/N$, as in the classical random graph process. Now though, to each vertex we associate an independent Poisson process with rate $\lambda(N)$, and we interpret points of each Poisson process as times when the corresponding vertex is *struck by lightning*. When a vertex is struck by lightning, all the *edges* in that vertex's current component are removed, reducing the component to a collection of isolated vertices.

So, in contrast to frozen percolation, the number of vertices is preserved in the forest fire process, at the expense of monotonicity. Ráth and Tóth assume again that the lightning rate is chosen such that $1/N \ll \lambda(N) \ll 1$, for identical reasons to frozen percolation. That is, small components are 'never' affected by lightning, while giant components are 'immediately' broken into singleton vertices. Again, Ráth and Tóth consider

$$v_k^N(t) := \frac{1}{N}\#\{\text{vertices in size } k \text{ components at time } t\}, \quad k \geq 1,\, t \geq 0.$$

They introduce the following *modified Smoluchowski equations* to approximate the evolution of $(v^N(\cdot))$ when N is large.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_k(t) = \frac{k}{2} \sum_{\ell=1}^{k-1} v_\ell(t) v_{k-\ell}(t) - k v_k(t), \quad k \geq 2, \tag{1.22}$$

$$\sum_{k\geq 1} v_k(t) = 1, \quad t \geq 0.$$

It is important to note that these equations do not directly describe the evolution of $v_1(\cdot)$. Instead, the second equation controls v_1 implicitly, by fixing the total mass to be constant. In particular, unlike the Flory equations (1.6), it is not possible to solve (1.22) for each k by induction.

Ráth and Tóth show that whenever $\sum k^3 v_k(0) < \infty$, there exists a unique solution to the modified Smoluchowski equations (1.22). Their argument involves showing that an exponential generating function corresponding to $(v_k(t))$ satisfies a version of the Burgers control problem. They also show a convergence result for the discrete processes, which is improved slightly by Crane, Freeman and Tóth [18]. The latter authors show that when $(v^N(\cdot))$ corresponds to a sequence of mean-field forest fire processes and satisfies $v_k^N(0) \to v_k(0)$ for each $k \in \mathbb{N}$ as $N \to \infty$, and where $v(0)$ satisfies $\sum k^3 v_k(0) < \infty$, then for every $T > 0$,

$$\sup_{k\in\mathbb{N}} \sup_{t\in[0,T]} |v_k^N(t) - v_k(t)| \stackrel{d}{\to} 0,$$

as $N \to \infty$, where $v(\cdot)$ is the unique solution to (1.22).

As Ráth and Tóth explain, the modified Smoluchowski equations (1.22) admit a *stationary* solution

$$v_k(\infty) := \frac{2}{k}\binom{2k-2}{k-1}4^{-k}. \tag{1.23}$$

However, the distribution in (1.23) does not satisfy the third-moment condition $\sum_{k\geq 1} k^3 v_k(\infty) < \infty$, and so, counterintuitively, it is currently not known whether the stationary solution is the unique solution to (1.22) with these initial conditions. In addition, it is plausible that any solution to the modified Smoluchowski equations converges as $t \to \infty$ to $v(\infty)$, but this is also open.

Settling these questions motivates the model of frozen percolation with k types considered in Chapter 5. We will explain this relation in more detail in Section 5.1.3.

Chapter 2

Critical random forests

In this self-contained chapter, we review Aldous's results [5] about the distribution of the sequence of component-sizes in $G(N,p)$, when $p = p(N)$ lies in the *critical window* $p(N) = \frac{1+\lambda N^{-1/3}}{N}$. Aldous describes the scaling limit for the largest such components in terms of the excursions of a reflected Brownian motion with time-dependent drift. We prove a similar result for the random forest obtained by conditioning $G(N,p)$ to have no cycles, for the same range of p. We describe a scaling limit for the largest components of such a *critical random forest*, but now using a reflected diffusion whose drift is space-dependent as well as time-dependent.

2.1 Background

2.1.1 The critical window

In Section 1.1.1, we discussed the phase transition of the sparse Erdős–Rényi random graph $G(N,c/N)$ at $c = 1$. As with many phase transitions, the asymptotic behaviour for $c = 1$ is qualitatively different from the asymptotic behaviour both for $c < 1$ and for $c > 1$. By looking at finer scalings for which $Np(N) \to 1$, we can examine exactly how this transition from a graph with logarithmic components to a graph with a giant component takes place.

- When $Np(N) \to 1$ and $N^{1/3}[Np(N)-1] \to -\infty$, Bollobás [13], who later calls this the *barely subcritical* regime, shows that the largest component has size $(1+o(1))\frac{2\log(\epsilon^3 N)}{\epsilon^2}$ with high probability, where $\epsilon(N) = 1 - Np(N)$.
- Bollobás [13] also treats the *barely supercritical* regime where $Np(N) \to 1$ and $\frac{N^{1/3}}{\log N}[Np(N)-1] \to +\infty$. In this case, the largest component has size $2\epsilon(1+o(1))$ with high probability, where $\epsilon(N) = Np(N) - 1$. Furthermore, the ratio between the size of the largest component and the size of the second-largest component is asymptotically infinite. Łuczak [44] shows the same result for the extended range $N^{1/3}[Np(N)-1] \to +\infty$.
- In [44], Łuczak studies further the regime between these subcritical and supercritical behaviours, and establishes the precise scaling range of the *critical window*, $p(N) = \frac{1+\lambda N^{-1/3}}{N}$ for $\lambda \in \mathbb{R}$. For such p, $G(N,p)$ shares the property of $G(N, 1/N)$ that for each fixed $k \in \mathbb{N}$, the kth largest component has size $\Theta_{\mathrm{p}}(N^{2/3})$. However, as we shall see shortly, different values of $\lambda \in \mathbb{R}$ lead to different asymptotic distributions for the component-size sequence.

2.1.2 Exploring a random graph in the critical window

Recall the definition of the breadth-first exploration process $(S_m)_{m\geq 0}$, and the application to $G(N,p)$ introduced in Section 1.2.2. The key observation is that the increments of this exploration process are binomial random variables. Recall from (1.17) that, conditional on $(Z_0, Z_1, \dots, Z_m)$,

$$S_m - S_{m-1} \overset{d}{=} \mathrm{Bin}(N - m - (Z_m \vee 1), p) - 1. \tag{2.1}$$

Take $(S_m^{N,\lambda})_{m\geq 0}$ to be the exploration process, and $(Z_m^{N,\lambda})_{m\geq 0}$ the reflected exploration process associated to the random graph $G(N, \frac{1+\lambda N^{-1/3}}{N})$, and define the rescaled exploration process and reflected exploration process respectively as

$$\tilde{S}^{N,\lambda}(s) := N^{-1/3} S^{N,\lambda}_{\lfloor N^{2/3}s \rfloor}, \quad \tilde{Z}^{N,\lambda}(s) := N^{-1/3} Z^{N,\lambda}_{\lfloor N^{2/3}s \rfloor}. \tag{2.2}$$

It can be shown that when both N and m are large, with high probability $Z_m \ll m$. Therefore, when $m = sN^{2/3}$, the expectation of the increment in (2.1) is approximately $(\lambda - s)N^{-1/3}$. The motivates considering a Brownian motion with drift $\lambda - s$ at time s.

Definition 2.1. For fixed $\lambda \in \mathbb{R}$, let W be a standard Brownian motion and let

$$W^\lambda(s) := W(s) + \lambda s - \tfrac{1}{2}s^2.$$

We call W^λ a *parabolically-drifting* Brownian motion. Then define

$$B^\lambda(s) := W^\lambda(s) - \min_{s' \in [0,s]} W^\lambda(s'), \tag{2.3}$$

to be the corresponding *reflected parabolically-drifting* Brownian motion. We also define $C_1^\lambda \geq C_2^\lambda \geq \ldots$ to be the lengths of the excursions of B^λ above zero, arranged in decreasing order.

Let $C_1^{N,\lambda} \geq C_2^{N,\lambda} \geq \ldots$ be the sizes of the components of $G(N, \frac{1+\lambda N^{-1/3}}{N})$, in decreasing order. Aldous's main result is:

THEOREM 2.2. [5, Theorem 2] For any $\lambda \in \mathbb{R}$, the convergence

$$N^{-2/3}(C_1^{N,\lambda}, C_2^{N,\lambda}, \ldots) \quad \stackrel{d}{\to} \quad (C_1^\lambda, C_2^\lambda, \ldots),$$

holds as $N \to \infty$ with respect to the $\ell^2_\searrow$ topology.

The main ingredient is the following:

THEOREM 2.3. [5, Theorem 3] For any $\lambda \in \mathbb{R}$, $\tilde{S}^{N,\lambda} \to W^\lambda$ uniformly on compact intervals in distribution.

2.1.3 Critical forests and results

Definition 2.4. Let $\mathcal{F}_\mathcal{A}$ be the set of forests with vertex set $\mathcal{A}$. We will use the shorthand $\mathcal{F}_N$ for the set of forests on $[N]$. In this chapter, we consider $\bar{G}(N,p)$, an $\mathcal{F}_N$-valued random variable, given for $N \in \mathbb{N}$ and $p \in [0,1)$ by conditioning $G(N,p)$ on the event that it contains no cycles. We call $\bar{G}(N,p)$ an *acyclic random graph*, and it is an example of a *random forest*.

From now on, we take $(S_m^{N,\lambda})_{m\geq 0}$ to be the following exploration process of $\bar{G}\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$. To make proofs easier, it will be convenient if the sequence $(v_1, v_2, \dots, v_N)$ of vertices in exploration order is breadth-first, as introduced in Section 1.2.1, but with each set of children in random order. That is, we choose v_1 uniformly at random from $[N]$. Then if v_1 has exactly k neighbours, we let $v_2, \dots, v_{k+1}$ be these neighbours *in uniformly random order*. Now proceed similarly for the neighbours of v_2, and so on, choosing the next vertex uniformly at random from those that remain whenever a component is exhausted.

We define $(Z_m^{N,\lambda})_{m\geq 0}$ to be the corresponding reflected exploration process.

The goal of this chapter is to derive a result similar to Theorem 2.2 for $\bar{G}\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$.

A reflected SDE

We define the following function

$$g(x) = \frac{1}{\pi}\int_0^\infty \exp(-\tfrac{4}{3}t^{3/2})\cos(xt + \tfrac{4}{3}t^{3/2})\mathrm{d}t, \tag{2.4}$$

which is the density of a stable distribution with parameter 3/2 that we will introduce in more detail in Section 2.1.4. Then, we define

$$\alpha(b,\lambda) := \frac{\int_0^\infty a^{-1/2} g(\lambda - a)\exp\Big(\frac{(\lambda-a)^3}{6}\Big)\exp(-\frac{b^2}{2a})\mathrm{d}a}{\int_0^\infty a^{-3/2} g(\lambda - a)\exp\Big(\frac{(\lambda-a)^3}{6}\Big)\exp(-\frac{b^2}{2a})\mathrm{d}a}, \quad b > 0,\ \lambda \in \mathbb{R}. \tag{2.5}$$

Lemma 2.5. The function g defined in (2.4) is positive, bounded, uniformly continuous, and satisfies $g(x) \to 0$ as $x \to \pm\infty$. Furthermore, the function α is well-defined, and increasing in its first argument, and satisfies $\alpha(b,\lambda) \to 0$ as $b \downarrow 0$, uniformly on λ in compact intervals.

We prove all these properties as part of Lemmas 2.22, 2.23, 2.25, and Proposition 2.26 in Section 2.2.6. This function $\alpha(b,\lambda)$ will be the additive correction one must make to the drift of the rescaled exploration process in order to account for the condition of acyclicity, when the height is b.

Proposition 2.6. Consider a standard Brownian motion $W(\cdot)$ with natural filtration $\mathcal{F}^W$. For each $\lambda \in \mathbb{R}$, there exists a unique pair of $\mathcal{F}^W$-adapted non-negative processes Z^λ, K^λ satisfying:

$$\begin{cases} Z^\lambda(0) = 0, \\ \mathrm{d}Z^\lambda(t) = \Big[\lambda - t - \alpha\Big(Z^\lambda(t), \lambda - t\Big)\Big]\mathrm{d}t + \mathrm{d}W(t) + K^\lambda(t), \end{cases} \tag{2.6}$$

where $K^\lambda(\cdot)$ is continuous and increasing, with $K^\lambda(0) = 0$, and $\int_0^\infty Z^\lambda(s)\mathrm{d}K^\lambda(s) = 0$.

This result will be proved in Section 2.2.7, and relies on a local Lipschitz property for α which is established in Section 2.2.6. We say Z^λ is a solution to the *reflected SDE* (2.6).

Convergence

Recall that $Z^{N,\lambda}$ is the reflected exploration process of $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$. For $s \geq 0$ let

$$\tilde{Z}_s^{N,\lambda} := N^{-1/3} Z^{N,\lambda}_{\lfloor N^{2/3} s \rfloor}.$$

From now on, we fix $\lambda \in \mathbb{R}$ and let $C_1^\lambda \geq C_2^\lambda \geq \dots$ be the lengths of the excursions of Z^λ above zero, arranged in decreasing order. Also, let $C_1^{N,\lambda} \geq C_2^{N,\lambda} \geq \dots$ be the sizes of the components of the graph $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$, in decreasing order. We define $\ell^2_\searrow$ to be the set of non-increasing sequences equipped with the ℓ_2-topology. Our main result is

THEOREM 2.7. The convergence

$$N^{-2/3}(C_1^{N,\lambda}, C_2^{N,\lambda}, \dots) \quad \xrightarrow{d} \quad (C_1^\lambda, C_2^\lambda, \dots), \tag{2.7}$$

holds as $N \to \infty$ in $\ell^2_\searrow$.

The main ingredient is the following convergence result for the rescaled exploration processes, analogous to Aldous's Theorem 2.3.

THEOREM 2.8. For each $\lambda \in \mathbb{R}$, we have

$$\tilde{Z}^{N,\lambda} \quad \stackrel{d}{\to} \quad Z^{\lambda}, \tag{2.8}$$

uniformly on compact time-intervals.

The proof of Theorem 2.8 is given in Section 2.2. A few technical details, mostly related to the fact that α is not globally Lipschitz, are treated in Sections 2.2.6 and 2.2.7. We will proceed by showing that the increments of these discrete (reflected) exploration processes have expectation and variance which match asymptotically the coefficients of (2.6). Convergence of the drift is the main challenge, and we require a technically-involved calculation to treat a binomial distribution tilted by a sequence enumerating a class of weighted forests.

Theorem 2.7 follows from Theorem 2.8, but a more technical argument is required here than for unreflected exploration processes. We will use the fact that the components of $\bar{G}(N,p)$ are, conditional on their size, uniform trees. This argument occupies Section 2.3.

2.1.4 Acyclic random graphs and enumerating forests

We will make particular use of a result of Britikov [16] that provides asymptotics for $f(N,m)$, the number of forests on $[N]$ with exactly m edges. Such a forest has exactly $N-m$ trees, and so

$$f(N,m) = \frac{N!}{(N-m)!} \sum_{\substack{k_1+\ldots+k_{N-m}=N \\ k_i \geq 1}} \prod_{j=1}^{N-m} \frac{k_j^{k_j-2}}{k_j!},$$

which suggests that an argument using generating functions and Bell polynomials will be applicable. (A comprehensive general introduction to such methods can be found in Chapter 1 of Pitman's notes [59].) Britikov defines a random variable ξ with a distribution parameterised by a constant $x \in \mathbb{R}_+$ such that

$$\mathbb{P}(\xi = k) \propto \frac{k^{k-2}x^k}{k!}, \quad \text{with normalising constant } B(x) = \sum_{k \geq 1} \frac{k^{k-2}x^k}{k!}.$$

This normalising constant $B(x)$ is finite precisely when $x \leq 1/e$. Taking $\xi_1, \xi_2, \ldots$ to be IID copies of ξ, we obtain

$$f(N,m) = \frac{N!}{(N-m)!} \frac{B(x)^{N-m}}{x^N} \mathbb{P}(\xi_1 + \ldots + \xi_{N-m} = N).$$

Thus laws of large numbers for ξ give asymptotics for $f(N,m)$. For the range of m we will be interested in, it is most relevant to consider $x = 1/e$, for which ξ is in the domain of attraction of a particular stable law with parameter $3/2$. The density of this stable law is $g(x)$, as defined in (2.4). The relevant regime of Britikov's result is summarised by Łuczak and Pittel in [45]:

Lemma 2.9. [45, Lemma 2.1.ii] For any constant $c > 0$, as $N \to \infty$,

$$f(N,m) = (1 + o(1)) \frac{\sqrt{2\pi} N^{N-1/6}}{2^{N-m}(N-m)!} g\left(\frac{2m - N}{N^{2/3}}\right), \tag{2.9}$$

uniformly for m satisfying $|2m - N|^3/N^2 \leq c$.

As we shall see in the proof of our main result, it follows from this that the asymptotic probability that $G(N,p)$ is acyclic in this regime is $\Theta(N^{-1/6})$. A precise statement and proof appears as Lemma 2.17 to follow.

2.1.5 Preliminary results

Before starting the main proof, we state a lemma providing a useful relation between $G(N,p)$ and $\bar{G}(N,p)$, which we will use repeatedly throughout the chapter.

Lemma 2.10. For all $N \in \mathbb{N}$, $p \in [0,1)$, there exists a coupling of $G(N,p)$ and $\bar{G}(N,p)$ such that $E(\bar{G}(N,p)) \subseteq E(G(N,p))$ almost surely.

Proof. Let $\mathbb{P}_{N,p}$ and $\bar{\mathbb{P}}_{N,p}$ be the laws of $G(N,p)$ and $\bar{G}(N,p)$ respectively, on the set of graphs with vertex set $[N]$, which is equivalent to $\{0,1\}^{\binom{[N]}{2}}$. By Strassen's theorem [65], it suffices to show that $\bar{\mathbb{P}}_{N,p}(B) \leq \mathbb{P}_{N,p}(B)$ for all increasing events $B \subseteq \{0,1\}^{\binom{[N]}{2}}$. Since $\mathbb{P}_{N,p}$ is product measure on the set of possible edges, the Harris inequality [31] applies. In this setting, the most useful statement of this result is

$$\mathbb{P}_{N,p}(B \cap A^c) \geq \mathbb{P}_{N,p}(B)\mathbb{P}_{N,p}(A^c),$$

for A any decreasing event, and B any increasing event. Here, take A to be the decreasing event that the graph is acyclic. From this it follows directly that $\mathbb{P}_{N,p}(B|A) \leq \mathbb{P}_{N,p}(B)$, that is $\bar{\mathbb{P}}_{N,p}(B) \leq \mathbb{P}_{N,p}(B)$. □

Janson and Spencer [34] give another description of the limit of component sizes in the critical window for $G(N,p)$. The following result is a consequence of their Theorem 4.1.

Proposition 2.11. Fix $\lambda \in \mathbb{R}$. If $|C^{N,\lambda}(v)|$ is the size of the component containing a uniformly-chosen vertex v in $G\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$, then there exists $\Theta^\lambda \in (0,\infty)$ such that

$$N^{-1/3}\mathbb{E}\Big[|C^{N,\lambda}(v)|\Big] \to \Theta^\lambda,$$

as $N \to \infty$. Thus by Lemma 2.10, if we now let $|C^{N,\lambda}(v)|$ be the size of the component containing a uniformly-chosen vertex in the *conditioned* graph $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$, we have

$$\limsup_{N\to\infty} N^{-1/3}\mathbb{E}\Big[|C^{N,\lambda}(v)|\Big] \leq \Theta^\lambda. \tag{2.10}$$

Note. From the coupling (1.1), Θ^λ is increasing as a function of λ, and $\Theta^\lambda \to 0$ as $\lambda \to -\infty$.

2.2 Convergence of the reflected exploration process

We will show shortly that the reflected exploration process $Z^{N,\lambda}$ has the Markov property. To show Theorem 2.8, we must check that the expected increments of the rescaled process $\tilde{Z}^{N,\lambda}$ converge to the drift term in (2.6), uniformly in some sense.

Proposition 2.12. Fix $T, K < \infty$, and $\lambda \in \mathbb{R}$. Then, uniformly on $m \in [0, TN^{2/3}]$ and $r \in [1, \rho N^{1/3}]$,

$$N^{1/3}\mathbb{E}\Big[Z_{m+1}^{N,\lambda} - Z_m^{N,\lambda} \,|\, Z_m^{N,\lambda} = r\Big] - \Big[\lambda - \tfrac{m}{N^{2/3}} + \alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - \tfrac{m}{N^{2/3}}\Big)\Big] \quad \to \quad 0, \quad (2.11)$$

as $N \to \infty$.

The proof of this proposition is completed in Section 2.2.4, after some preliminary asymptotic calculations concerning forests in random graphs.

It is also necessary to establish the convergence of the variance of the rescaled increments, and regularity properties that ensure the limit process is continuous and does not stick at zero.

Proposition 2.13. For any $\delta > 0$ uniformly on $m \in [0, TN^{2/3}]$ and $r \in [1, \rho N^{1/3}]$,

$$\mathbb{E}\Big[\Big[Z_{m+1}^{N,\lambda} - Z_m^{N,\lambda}\Big]^2 \,\Big|\, Z_m^{N,\lambda} = r\Big] \to 1, \quad (2.12)$$

$$N^{2/3}\mathbb{P}\Big(|Z_{m+1}^{N,\lambda} - Z_m^{N,\lambda}| > \delta N^{1/3} \,\Big|\, Z_m^{N,\lambda} = r\Big) \to 0, \quad (2.13)$$

as $N \to \infty$. In addition,

$$\liminf_{N\to\infty} \inf_{m\in[0,TN^{2/3}]} \mathbb{E}\Big[\Big[Z_{m+1}^{N,\lambda}\Big]^2 \,\Big|\, Z_m^{N,\lambda} = 0\Big] > 0. \quad (2.14)$$

The proof of this proposition occupies Section 2.2.5.

In Section 2.2.7, we explain how this pair of propositions is sufficient for Theorem 2.8. The main ingredient will be Theorem 2.28, a special case from Stroock and Varadhan's very general results [66] on the convergence of Markov processes to reflected diffusions.

2.2.1 Stack forests

Definition 2.14. For a graph G, we say a set $A \subseteq V(G)$ is *separated* in G if no pair of vertices in A lie in the same component of G.

Recall from Section 2.1.3 that we are considering a breadth-first exploration process of $\bar{G}(N,p)$ with uniform ordering within each set of children. For the remainder of this short section, we will suppress the dependence on N and p from the notation for the exploration process, since the result to follow holds for all $p \in (0,1)$. Then $\mathcal{Z}_m$ is the *stack* of vertices which have been seen but not explored yet. Note that all the vertices in $\mathcal{Z}_m$ are in the same component of $\bar{G}(N,p)$, since components are explored one-by-one. In particular, in the graph restricted to $[N]\backslash\{v_1,\dots,v_m\}$, no pair of vertices in $\mathcal{Z}_m$ lie in the same component, as otherwise there would be a cycle in $\bar{G}(N,p)$. We refer to the Z_m trees on $[N]\backslash\{v_1,\dots,v_m\}$ containing each $v \in \mathcal{Z}_m$ as the *stack forest*, as in Figure 2.1. We can see that the vertices in $\mathcal{Z}_m$ are *separated* in the restricted graph on $[N]\backslash\{v_1,\dots,v_m\}$.

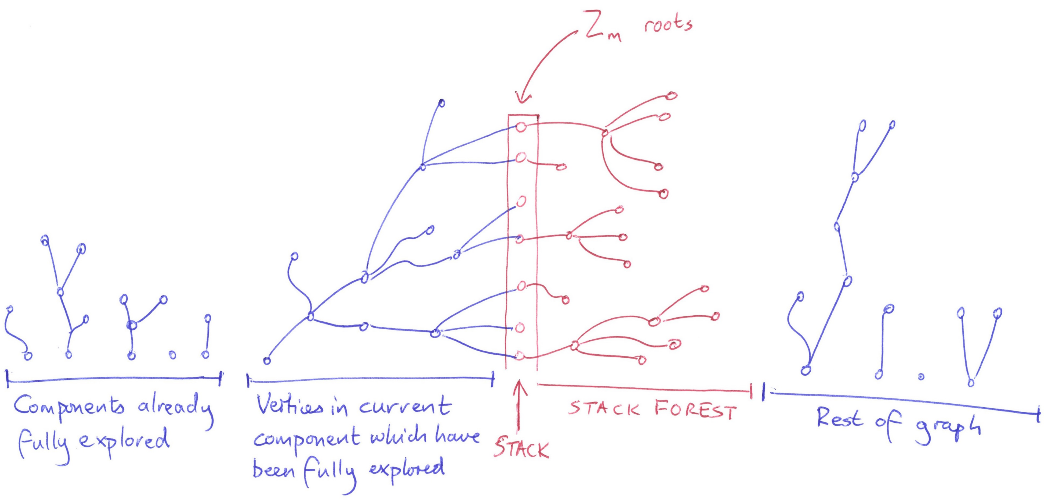

Fig. 2.1 Illustration of the definition of *stack forest*

Now, suppose we condition on $\{v_1,\dots,v_m\} \cup \mathcal{Z}_m$, and the structure of $\bar{G}(N,p)$ on these $m + Z_m$ vertices. Then, the graph restricted to $[N]\backslash\{v_1,\dots,v_n\}$ has the same distribution as

$$\bar{G}([N]\backslash\{v_1,\dots,v_n\},p),$$

with the extra condition that no pair of vertices from $\mathcal{Z}_m$ lie in the same component.

We expand this explanation considerably in the proof of the following lemma, which formalises the claim that $(Z_m)_{m\geq 0}$ is Markov, and characterises its transition probabilities via separation of the current stack in the remainder of the graph.

Lemma 2.15. For any $m \geq 0$, and $r \geq 1$,

$$\begin{aligned} &\mathbb{P}(Z_{m+1} - Z_m = \ell - 1 \,|\, Z_m = r, Z_{m-1} = r_{m-1}, \ldots, Z_1 = r_1) \qquad (2.15)\\ &\quad \propto \binom{N-m-r}{\ell} p^{\ell}(1-p)^{N-m-r-\ell}\mathbb{P}\Big([r+\ell-1] \text{ separated in } \bar{G}(N-m-1,p)\Big), \end{aligned}$$

for $\ell = 0, 1, \ldots, N-m-r$.

Proof. Because the vertices are exchangeable in both $\bar{G}(N,p)$ and the exploration process, the LHS of (2.15) is unchanged by conditioning further on which vertices are seen during the initial phase of the exploration process. That is, if we define the event

$$\mathcal{B} := \Big\{ Z_m = r, Z_{m-1} = r_{m-1}, \ldots, Z_1 = r_1,\, (v_1, \ldots, v_m) = (1, \ldots, m), \\ \mathcal{Z}_m = \{m+1, \ldots, m+r\} \Big\}$$

then

$$\begin{aligned} &\mathbb{P}(Z_{m+1} - Z_m = \ell - 1 \,|\, Z_m = r, Z_{m-1} = r_{m-1}, \ldots, Z_1 = r_1) \qquad (2.16)\\ &\quad = \mathbb{P}(Z_{m+1} - Z_m = \ell - 1 \,|\, \mathcal{B}). \end{aligned}$$

Note that $\mathcal{B}$ is defined in terms of $(r_1, \ldots, r_{m-1}, r)$. Also note that since $\bar{G}(N,p)$ is acyclic, this richer conditioning exactly specifies the neighbourhoods of vertices $1, \ldots, m$. That is,

$$\mathcal{B} \quad \Rightarrow \quad \Gamma(i) = A_i, \quad \forall i \in [m], \qquad (2.17)$$

where each $A_i \subseteq [m+r]$ is also a function of $(r_1, \ldots, r_{m-1}, r)$. Furthermore, on $\mathcal{B}$, the number of edges in the graph incident to at least one vertex in $[m]$ is a constant, say $e_{[m]}$, depending on $(r_1, \ldots, r_{m-1}, r)$. Obviously, the converse direction of (2.17) is generally false since there are extra sources of randomness in the construction of the exploration process. However, we can show the Markov property by restricting attention to those forests on $[N]$ for which the conclusion of (2.17) holds.

Let $\mathbb{A}_N$ be the set of forests H on $[N]$ for which $\Gamma_H(i) = A_i$ for $i \in [m]$. Then, we claim that

$$\mathbb{P}\Big((v_1, \dots, v_m) = (1, \dots, m) \,|\, \bar{G}(N,p) = H\Big) \quad \text{is the same for every } H \in \mathbb{A}_N.$$

This claim follows because, in turn, each v_i depends only on a uniform choice from the remaining $N - i + 1$ vertices in the graph (in the case of starting a new component), or a uniform choice of orderings of $|\Gamma_H(j)|$ neighbours, for some $j < i$ (otherwise).

Consider any $H \in \mathbb{A}_N$. Given that $\bar{G}(N,p) = H$, the event $\mathcal{B}$ holds if and only if $(v_1, \dots, v_m) = (1, \dots, m)$. Therefore

$$\mathbb{P}\Big(\mathcal{B} \,|\, \bar{G}(N,p) = H\Big) \quad \text{is the same for every } H \in \mathbb{A}_N. \tag{2.18}$$

Now let $\mathbb{A}'_N$ be the set of forests H' on $[N]\backslash[m]$ for which $\{m+1, \dots, m+r\}$ are separated. There is a bijection $\mathbb{A}_N \to \mathbb{A}'_N$ given by restricting the vertex set. Note that, while $\mathbb{A}_N$ depends on the whole sequence $(r_1, \dots, r_{m-1}, r)$, the restricted set $\mathbb{A}'_N$ depends only on r. So

$$\begin{aligned}
&\mathbb{P}(Z_{m+1} - Z_m = \ell - 1 \,|\, \mathcal{B}) \\
&\quad \propto \sum_{H \in \mathbb{A}_N} \mathbb{P}\Big(\mathcal{B} \,|\, \bar{G}(N,p) = H\Big) \mathbb{P}\Big(\bar{G}(N,p) = H\Big) \mathbb{1}\{\deg_{H|_{[N]\backslash[m]}}(m+1) = \ell\} \\
&\quad \overset{(2.18)}{\propto} \sum_{H \in \mathbb{A}_N} \mathbb{P}\Big(\bar{G}(N,p) = H\Big) \mathbb{1}\{\deg_{H|_{[N]\backslash[m]}}(m+1) = \ell\} \\
&\quad \propto \sum_{H \in \mathbb{A}_N} p^{|E(H)|}(1-p)^{\left|\binom{[N]}{2}\backslash E(H)\right|} \mathbb{1}\{\deg_{H|_{[N]\backslash[m]}}(m+1) = \ell\} \\
&\quad \propto p^{e_{[m]}}(1-p)^{Nm - \binom{m}{2} - e_{[m]}} \\
&\qquad \times \sum_{H' \in \mathbb{A}'_N} p^{|E(H')|}(1-p)^{\left|\binom{[N]\backslash[m]}{2}\backslash E(H')\right|} \mathbb{1}\{\deg_{H'}(m+1) = \ell\} \\
&\quad \propto \sum_{H' \in \mathbb{A}'_N} p^{|E(H')|}(1-p)^{\left|\binom{[N]\backslash[m]}{2}\backslash E(H')\right|} \mathbb{1}\{\deg_{H'}(m+1) = \ell\}.
\end{aligned}$$

Note that $H' \in \mathbb{A}'_N$ implies that $\Gamma_{H'}(m+1) \cap \{m+2, \dots, m+r\} = \varnothing$. Then, exchangeability of the vertices $\{m+r+1, \dots, N\}$ allows us to consider a specific neighbourhood of vertex $m+1$ in H', rather than merely its degree.

$$\mathbb{P}\big(Z_{m+1} - Z_m = \ell - 1 \,|\, \mathcal{B}\big) \propto \binom{N-m-r}{\ell}$$
$$\times \sum_{H' \in \mathbb{A}'_N} p^{|E(H')|}(1-p)^{\left|\binom{[N]\setminus[m]}{2}\setminus E(H')\right|} \mathbb{1}\{\Gamma_{H'}(m+1) = \{m+r+1,\ldots,m+r+\ell\}\}.$$

Now, simply by removing vertex $m+1$, there is a bijection between the set of forests $H' \in \mathbb{A}'_N$ for which $\Gamma_{H'}(m+1) = \{m+r+1,\ldots,m+r+\ell\}$ and the set of forests on $[N]\setminus[m+1]$, for which $\{m+2,\ldots,m+r+\ell\}$ are separated. Recall from Definition 2.4 the notation $\mathcal{F}_{[N]\setminus[m+1]}$ for the set of forests on $[N]\setminus[m+1]$. We then have

$$\mathbb{P}\big(Z_{m+1} - Z_m = \ell - 1 \,|\, \mathcal{B}\big) \propto \binom{N-m-r}{\ell}\left(\frac{p}{1-p}\right)^{\ell}$$

$$\times \sum_{F \in \mathcal{F}_{[N]\setminus[m+1]}} p^{|E(F)|}(1-p)^{\left|\binom{[N]\setminus[m+1]}{2}\setminus E(F)\right|} \mathbb{1}\{\{m+2,\ldots,\ldots,m+r+\ell\} \text{ separated in } F\}.$$

Then, considering the sum as the probability of an event in the weighted random forest on $N-m-1$ vertices, we obtain

$$\mathbb{P}\big(Z_{m+1} - Z_m = \ell - 1 \,|\, \mathcal{B}\big) \propto \binom{N-m-r}{\ell} p^{\ell}(1-p)^{N-m-r-\ell}$$

$$\mathbb{P}\Big([r+\ell-1] \text{ separated in } \bar{G}(N-m-1,p)\Big),$$

and so the required statement follows using (2.16). □

We want to quantify exactly how large a probabilistic penalty is incurred by adding an extra vertex to the stack, and so will consider limits of the quantity

$$\frac{\mathbb{P}\Big([r+\ell] \text{ separated in } \bar{G}(N-m-1,p)\Big)}{\mathbb{P}\Big([r+\ell-1] \text{ separated in } \bar{G}(N-m-1,p)\Big)}.$$

Given a graph in which $[r+\ell-1]$ are separated, the conditional probability that $r+\ell$ is also separated depends on the size of the stack forest rooted by $[r+\ell-1]$. So we will calculate the expected size of a stack forest in Section 2.2.3. It will be useful to have precise asymptotics for the probability that $G(N,p)$ is acyclic, which we derive in

Section 2.2.2. We then use this to calculate the probability that the stack forest has a particular size.

2.2.2 Enumerating weighted stack forests

In this first section, we consider the probability that $G(N,p)$ is acyclic. Recall $f(N,m)$ is the number of forests with vertex set $[N]$ and exactly m edges. Therefore

$$\mathbb{P}(G(N,p) \text{ acyclic}) = (1-p)^{\binom{N}{2}} \sum_{m=0}^{N-1} f(N,m) \Big(\frac{p}{1-p}\Big)^m. \tag{2.19}$$

We call this quantity $F(N,p)$.

Lemma 2.16. For any $N \geq 0$ and any $p \in (0,1)$,

$$F(N,p) \geq F(N+1,p) \geq F(N,p)\Big[1 - \frac{1}{2} N p^2 \mathbb{E}\Big[|C^{N,p}(v)|\Big]\Big], \tag{2.20}$$

where $C^{N,p}(v)$ is the component containing a uniformly-chosen vertex v in $G(N,p)$.

Proof. Graphs with zero, one or two vertices are certainly acyclic, so $F(0,p) = F(1,p) = F(2,p) = 1$, the statement is true for $N = 0,1$. We assume from now on that $N \geq 2$. We can define a forest on $[N+1]$ via the restriction to $[N]$ (which is clearly also a forest) and the neighbourhood of vertex $N+1$, where the latter must obey some conditions to avoid cycles. We take $\mathbb{P}$ to be a probability distribution which couples $G(N,p)$ and $G(N+1,p)$ such that $E(G(N,p)) \subseteq E(G(N+1,p))$, $\mathbb{P}$-a.s. Recall that in a graph G, for $v \in V(G)$, $\Gamma(v)$ is the set of vertices connected to v by an edge in $E(G)$. Then

$$F(N+1,p) = F(N,p)\mathbb{P}\big(\Gamma(N+1) \text{ separated in } G(N,p) \,|\, G(N,p) \text{ acyclic}\big),$$

and so the first inequality in (2.20) certainly holds. Now, for any set $A \subseteq [N]$, the event that A is separated in G is decreasing, while the event that G is acyclic is also decreasing. So, again by the Harris inequality,

$$F(N+1,p) \geq F(N,p)\mathbb{P}(\Gamma(N+1) \text{ separated in } G(N,p)),$$

and so

$$1 - \frac{F(N+1,p)}{F(N,p)} \leq \mathbb{P}(\Gamma(N+1) \text{ not separated in } G(N,p)). \qquad (2.21)$$

Observe that the event that $\Gamma(N+1)$ is not separated in $G(N,p)$ is the union over $i, j \in [N]$ of the events

$$\{i, j \text{ both in } \Gamma(N+1) \text{ and both in the same component of } G(N,p)\}.$$

Thus, by exchangeability of the vertices in $[N]$,

$$\mathbb{P}(\Gamma(N+1) \text{ not separated in } G(N,p)) \leq \binom{N}{2} p^2\, \mathbb{P}(1 \text{ and } 2 \text{ in same component of } G(N,p)).$$

Then, if $|C^{N,p}(1)|$ is the size of the component of $G(N,p)$ containing vertex 1,

$$\mathbb{P}(1 \text{ and } 2 \text{ in same component of } G(N,p)) = \frac{\mathbb{E}\Big[|C^{N,p}(1)|\Big] - 1}{N-1}.$$

We conclude that

$$\mathbb{P}(\Gamma(N+1) \text{ not separated in } G(N,p)) \leq \binom{N}{2} p^2 \cdot \frac{\mathbb{E}\Big[|C^{N,p}(1)|\Big] - 1}{N-1} \leq \frac{1}{2} N p^2 \mathbb{E}\Big[|C^{N,p}(1)|\Big],$$

from which the result follows, using (2.21) and the fact that the vertices in $G(N,p)$ are exchangeable. □

Recall the definition (2.19):

$$F(N,p) := \mathbb{P}(G(N,p) \text{ acyclic}) = (1-p)^{\binom{N}{2}} \sum_{m=0}^{N-1} f(N,m) \left(\frac{p}{1-p}\right)^m.$$

Now, using the asymptotics for $f(N,m)$ in (2.9), we obtain asymptotics for $F(N,p)$. Here, and in subsequent sections, some rather involved calculations are required, and in some places, various expansions have to be taken to fifth order. To avoid breaking the flow of the main argument, we postpone several detailed proofs, including for the following lemma, until Section 2.4.

Lemma 2.17. Fix $\lambda^- < \lambda^+ \in \mathbb{R}$. Given $p \in (0,1)$, let $\lambda = \lambda(N,p) = N^{1/3}(Np-1)$. Then

$$\mathbb{P}(G(N,p) \text{ acyclic}) = (1+o(1))g(\lambda)e^{3/4}\sqrt{2\pi}N^{-1/6}, \tag{2.22}$$

uniformly for $\lambda \in [\lambda^-, \lambda^+]$ as $N \to \infty$.

Motivated by the definition of stack forests, for each $0 \le r \le N$, let $\mathcal{A}_{N,r} \subseteq \mathcal{F}_N$ denote the set of forests where the vertices $1,\dots,r$ are separated. Furthermore, given a forest $F \in \mathcal{A}_{N,r}$, let $k_r(F)$ be the sum of the sizes of the components containing vertices $1,\dots,r$. We also define

$$\mathcal{A}_{N,r,k} := \{F \in \mathcal{A}_{N,r},\ k_r(F) = k\}, \tag{2.23}$$

the set of forests where $1,\dots,r$ are separated, and their stack forest has size k.

Definition 2.18. Given $p \in (0,1)$ and $N, N', r, k \in \mathbb{N}$ satisfying $N' \le N$, and $r \le k \le N$, we will use the following rescalings:

$$\lambda = \lambda(N,p) := N^{1/3}(Np-1), \quad a = a(N,k) := \frac{k}{N^{2/3}},$$

$$b = b(N,r) := \frac{r}{N^{1/3}}, \quad s = s(N,N') := \frac{N-N'}{N^{2/3}}. \tag{2.24}$$

For much of this and the following sections, it will be necessary to make estimates uniformly across several variables. For constants $T < \infty$, and $\lambda^- < \lambda^+$, and $0 < \epsilon < K < \infty$, we let

$$\Psi^N(\lambda^-, \lambda^+, \epsilon, K, T) := \Big\{(N', p, r, k) \in \mathbb{N} \times (0,1) \times \mathbb{N} \times \mathbb{N} \,:\, s(N,N') \in [0,T],$$

$$\lambda(N,p) \in [\lambda^-, \lambda^+],\ b(N,r) \in [\epsilon, K],\ k \in [r, KN^{2/3}]\Big\}.$$

We also define the following, which includes a broader range of r,

$$\bar{\Psi}^N(\lambda^-, \lambda^+, K, T) := \Big\{(N', p, r, k) \in \mathbb{N} \times (0,1) \times \mathbb{N} \times \mathbb{N} \,:\, s(N,N') \in [0,T],$$

$$\lambda(N,p) \in [\lambda^-, \lambda^+],\ r \in [1, KN^{1/3}],\ k \in [r, KN^{2/3}]\Big\}.$$

In addition, we define the projections of both of these sets into their first three entries

$$\Psi_0^N(\lambda^-,\lambda^+,\epsilon,K,T) := \Big\{(N',p,r)\,:\,s(N,N')\in[0,T],\,\lambda(N,p)\in[\lambda^-,\lambda^+],\,b(N,r)\in[\epsilon,K]\Big\}.$$

$$\bar\Psi_0^N(\lambda^-,\lambda^+,K,T) := \Big\{(N',p,r)\,:\,s(N,N')\in[0,T],\,\lambda(N,p)\in[\lambda^-,\lambda^+],\,r\in[1,KN^{1/3}]\Big\}.$$

The following lemma gives uniform asymptotics for the probability that $G(N',p)$ lies in $\mathcal{A}_{N',r,k}$. The proof is postponed until Section 2.4.2.

Lemma 2.19. Fix constants $\lambda^-,\lambda^+,\epsilon,K,T$ as in Definition 2.18. Then,

$$\mathbb{P}(G(N',p)\in\mathcal{A}_{N',r,k}) = (1+o(1))e^{3/4}g(\lambda-s-a)N^{-5/6}ba^{-3/2} \tag{2.25}$$

$$\times\exp\Big(-b(\lambda-s)-\tfrac{b^2}{2a}+\tfrac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\Big),$$

uniformly on $(N',p,r,k)\in\Psi^N(\lambda^-,\lambda^+,\epsilon,K,T)$, as $N\to\infty$.

2.2.3 Expected size of the stack forest

We now condition on $[r]$ being separated in $\bar G(N',p)$, and obtain an estimate on the expected size of the corresponding stack forest. Recall from (2.24) the definitions of $b=b(N,r)$ and $s=s(N,N')$, the rescaled stack size, and the graph vertex deficit count, respectively.

Lemma 2.20. Fix constants λ^-,λ^+,K,T as in Definition 2.18. Then,

$$N^{-2/3}\mathbb{E}\Big[k_r(\bar G(N',p))\,|\,\bar G(N',p)\in\mathcal{A}_{N',r}\Big]-\alpha(b,\lambda-s)\to 0, \tag{2.26}$$

uniformly on $(N',p,r)\in\bar\Psi_0^N(\lambda^-,\lambda^+,K,T)$, as $N\to\infty$.

Proof. We can rewrite the expectation in (2.26) in terms of the unconditioned random graphs $G(N',p)$ as follows.

$$\mathbb{E}\Big[k_r(\bar G(N',p))\,|\,\bar G(N',p)\in\mathcal{A}_{N',r}\Big] = \frac{\sum_{k=r}^{N'}k\mathbb{P}\Big(\bar G(N',p)\in\mathcal{A}_{N',r,k}\Big)}{\sum_{k=r}^{N'}\mathbb{P}\Big(\bar G(N',p)\in\mathcal{A}_{N',r,k}\Big)}$$

$$= \frac{\sum_{k=r}^{N'} k\mathbb{P}(G(N',p) \in \mathcal{A}_{N',r,k})}{\sum_{k=r}^{N'} \mathbb{P}(G(N',p) \in \mathcal{A}_{N',r,k})}. \tag{2.27}$$

We shall see that both of the sums in (2.27) are dominated by contributions from $k = \Theta(N^{2/3})$.

In order to use Lemma 2.19, we assume $\epsilon \in (0, K)$ is given. We will first show that (2.26) holds uniformly on $\Psi^N(\lambda^-, \lambda^+, \epsilon, K, T)$. Then, at the end, we will take $\epsilon \to 0$. We also select $M > K$, which we will take to ∞ shortly.

We write $h(a, b) := a^{-3/2}g(\lambda - s - a)\exp\left(\frac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\right)\exp(-b^2/2a)$. Since g is bounded, $h(a, b) \to 0$ as $a \to 0$ (indeed uniformly on $b \in [\epsilon, K]$, $\lambda \in \mathbb{R}$, $s \in \mathbb{R}_{\geq 0}$), so $\int_0^M h(a, b)\mathrm{d}a < \infty$ for all $M < \infty$. We observe that for all $\lambda \in \mathbb{R}$ and $a > 0$, we have $\exp\left(\frac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\right) \leq 1$. It also holds that

$$\inf_{\substack{a\in[0,K] \\ \lambda,s\in\mathbb{R}}} \frac{\partial}{\partial a}\left[\exp\left(\frac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\right)\right] = \inf_{\substack{a\in[0,K] \\ \lambda,s\in\mathbb{R}}} \left[-\frac{(\lambda-s-a)^2}{2}\right]\exp\left(\frac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\right) > -\infty.$$

The function h therefore has the following uniform continuity property. For any compact set $A \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ and for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, A) > 0$ such that whenever $(a, b), (a', b') \in A$ and $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$ then

$$\left|h(a, b) - h(a', b')\right| \leq \epsilon, \quad \forall\lambda \in \mathbb{R}, \forall s \in \mathbb{R}_{\geq 0}.$$

Furthermore, h is bounded away from zero on A, uniformly in λ and s. We may now use Lemma 2.19 to approximate every summand in (2.27), uniformly over the required range. (Recall from (2.24) that a is a linear function of k.) So

$$\sum_{k=r}^{\lceil MN^{2/3}\rceil} \mathbb{P}(G(N',p) \in \mathcal{A}_{N',r,k}) = (1 + o(1))bN^{-5/6}\exp\left(-b(\lambda - s) - \frac{(\lambda-s)^3}{6} + \frac{3}{4}\right)$$

$$\times N^{2/3}\int_0^M a^{-3/2}g(\lambda - s - a)\exp\left(\frac{(\lambda-s-a)^3}{6}\right)\exp\left(-\frac{b^2}{2a}\right)\mathrm{d}a,$$

$$\sum_{k=r}^{\lceil MN^{2/3}\rceil} k\mathbb{P}(G(N',p) \in \mathcal{A}_{N',r,k}) = (1 + o(1))bN^{-5/6}\exp\left(-b(\lambda - s) - \frac{(\lambda-s)^3}{6} + \frac{3}{4}\right)$$

$$\times N^{4/3} \int_0^M a^{-1/2} g(\lambda - s - a) \exp\Big(\tfrac{(\lambda-s-a)^3}{6}\Big) \exp\Big(-\tfrac{b^2}{2a}\Big) \mathrm{d}a, \tag{2.28}$$

uniformly on $(N', p, r) \in \Psi_0^N(\lambda^-, \lambda^+, \epsilon, K, T)$, as $N \to \infty$.

Observe, by comparison with the definition of α in (2.5), that

$$\lim_{M\to\infty} N^{-2/3} \frac{\sum_{k=r}^{\lceil MN^{2/3}\rceil} k\mathbb{P}(G(N', p) \in \mathcal{A}_{N',r,k})}{\sum_{k=r}^{\lceil MN^{2/3}\rceil} \mathbb{P}(G(N', p) \in \mathcal{A}_{N',r,k})} = (1 + o(1))\alpha(b, \lambda - s),$$

uniformly on $(N', p, r) \in \Psi_0^N(\lambda^-, \lambda^+, \epsilon, K, T)$.

Therefore, to apply (2.27) to verify (2.26), we must check that the contribution to the expectation from the event that the size of the stack forest is larger than $MN^{2/3}$ vanishes as $M \to \infty$. From (2.28), the contribution to the numerator of (2.27) from summands for which $k \in [r, \lceil MN^{2/3}\rceil]$ has order $N^{-5/6} \times N^{4/3} = N^{1/2}$. So to verify (2.26) uniformly on $\Psi_0^N(\lambda^-, \lambda^+, \epsilon, K, T)$, it will suffice to check that the following statement holds:

$$\lim_{M\to\infty} \limsup_{N\to\infty} \sup_{(N',p,r)\in\Psi_0^N(\lambda^-,\lambda^+,\epsilon,K,T)} N^{-1/2} \sum_{k=\lfloor MN^{2/3}\rfloor}^{N'} k\mathbb{P}(G(N', p) \in \mathcal{A}_{N',r,k}) = 0. \tag{2.29}$$

The stack forest is not too large

To show (2.29), we will show that the sequence $(k\mathbb{P}(G(N', p) \in \mathcal{A}_{N',r,k}))_{k\geq r}$ is eventually bounded by a geometric series. From the definition of $F(N, p)$ in (2.19), we have that

$$\mathbb{P}(G(N, p) \in \mathcal{A}_{N,r,k}) = (1 - p)^{\binom{N}{2} - \binom{N-k}{2}} \binom{N - r}{k - r} \left(\frac{p}{1 - p}\right)^{k-r} rk^{k-r-1} F(N - k, p). \tag{2.30}$$

An explanation of where each term in this expression comes from is given in the proof of Lemma 2.19 in Section 2.4.2. We will use this to control the ratio of the probabilities $\mathbb{P}(G(N', p) \in \mathcal{A}_{N',r,k})$ in the following lemma.

Lemma 2.21. Given the same constants as in Lemma 2.20, there exist constants $M < \infty$ and $\gamma > 0$ such that

$$\frac{(k+1)\mathbb{P}\big(G(N',p) \in \mathcal{A}_{N',r,k+1}\big)}{k\mathbb{P}\big(G(N',p) \in \mathcal{A}_{N',r,k}\big)} \le 1 - \gamma N^{-2/3}, \tag{2.31}$$

for large enough N, whenever $(N',p,r) \in \bar{\Psi}_0^N(\lambda^-, \lambda^+, K, T)$ and $k \in [MN^{2/3}, N'-1]$.

This lemma is proved in Section 2.4.3. But then, we can bound (2.29) via a geometric series as

$$N^{-1/2} \sum_{k=MN^{2/3}}^{N'} k\mathbb{P}\big(G(N',p) \in \mathcal{A}_{N',r,k}\big) \le N^{-1/2} \frac{\lceil MN^{2/3} \rceil \mathbb{P}\Big(G(N',p) \in \mathcal{A}_{N',r,\lceil MN^{2/3}\rceil}\Big)}{1 - (1 - \gamma N^{-2/3})}.$$

By Lemma 2.19, this RHS is

$$\begin{aligned}
&(1+o(1))N^{-1/2}\frac{1}{\gamma}N^{2/3} \cdot MN^{2/3} e^{3/4} g(\lambda - s - M) N^{-5/6} b M^{-3/2} \\
&\qquad \times \exp\Big(-b(\lambda - s) - \tfrac{b^2}{2M} + \tfrac{(\lambda-s-M)^3 - (\lambda-s)^3}{6}\Big) \\
&= (1+o(1)) M^{-1/2} e^{-b^2/2M} \exp\Big(\tfrac{(\lambda-s-M)^3-(\lambda-s)^3}{6}\Big) \times g(\lambda - s - M) \\
&\qquad \times \tfrac{e^{3/4}}{\gamma} b \exp\Big(-b(\lambda - s) - \tfrac{(\lambda-s)^3}{6}\Big).
\end{aligned}$$

Recall that g is uniformly bounded above and $\exp\Big(\frac{(\lambda-s-M)^3-(\lambda-s)^3}{6}\Big) \le 1$. Then observe that $M^{-1/2} e^{b^2/2M} \to 0$ as $M \to \infty$. Therefore

$$\lim_{M\to\infty} \limsup_{N\to\infty} \sup_{(N',p,r)\in\Psi_0^N(\lambda^-,\lambda^+,\epsilon,K,T)} N^{-1/2} \sum_{k=\lfloor MN^{2/3}\rfloor}^{N'} k\mathbb{P}\big(G(N',p) \in \mathcal{A}_{N',r,k}\big) = 0.$$

So we have finished the proof of (2.29), and thus we have shown that (2.26) holds uniformly on $\Psi_0^N(\lambda^-, \lambda^+, \epsilon, K, T)$.

Small stacks

To finish this proof of Lemma 2.20, it remains to extend the convergence to uniformity on $r \in [1, \lceil Kn^{1/3} \rceil]$, rather than on $[\lfloor \epsilon N^{1/3} \rfloor, \lceil KN^{1/3} \rceil]$.

Recall from Lemma 2.5 that $\alpha(b,\lambda)\to 0$ as $b\downarrow 0$ uniformly on compact intervals in λ. In particular

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\substack{\lambda\in[\lambda^-,\lambda^+]\\ s\in[0,T],\, r\in[1,\epsilon N^{1/3}]}}\alpha\Big(\tfrac{r}{N^{1/3}},\lambda-s\Big)=0. \tag{2.32}$$

Before Definition 2.18, we defined $k_r(F)$ for a forest F, but we can extend the definition to a general graph G with vertex set $[N]$. If $|C(i)|$ is the size of the component containing vertex $i\in[N]$, then let $k_r(G):=|C(1)|+\ldots+|C(r)|$, so some components may be counted at least twice. In particular, $k_r(G)$ is an increasing function of graphs. However, for any r, the set $\mathcal{A}_{N,r}$ is a decreasing family of graphs. Therefore

$$\mathbb{E}\big[k_r(G(N',p))\,|\,G(N',p)\in\mathcal{A}_{N',r}\big]\le\mathbb{E}\big[k_r(G(N',p))\big]\le r\mathbb{E}\Big[|C^{N',p}(1)|\Big], \tag{2.33}$$

where $|C^{N',p}(1)|$ is the size of the component containing vertex 1 in $G(N',p)$. From Proposition 2.11, for the range of N',p under consideration,

$$\limsup_{N\to\infty}\sup_{\substack{N'\in[N-TN^{2/3},N]\\ \lambda(N,p)\in[\lambda^-,\lambda^+]}}N^{-1/3}\mathbb{E}\Big[|C^{N',p}(1)|\Big]\le\Theta^{\lambda^+}<\infty. \tag{2.34}$$

We now take $r\le\epsilon N^{1/3}$ in (2.33), and apply (2.34) to obtain

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\substack{N'\in[N-TN^{2/3},N]\\ \lambda(N,p)\in[\lambda^-,\lambda^+]\\ r\in[1,\epsilon N^{1/3}]}}N^{-2/3}\mathbb{E}\big[k_r(G(N',p))\,|\,G(N',p)\in\mathcal{A}_{N',r}\big]\le\lim_{\epsilon\to 0}\epsilon\Theta^{\lambda^+}=0.$$

So, with (2.32), this gives

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\substack{N'\in[N-TN^{2/3},N]\\ \lambda(N,p)\in[\lambda^-,\lambda^+]\\ r\in[1,\epsilon N^{1/3}]}}\Big|N^{-2/3}\mathbb{E}\big[k_r(G(N',p))\,|\,G(N',p)\in\mathcal{A}_{N',r}\big]-\alpha\Big(\tfrac{r}{N^{1/3}},\lambda-s\Big)\Big|=0. \tag{2.35}$$

We already know that (2.26) holds uniformly on $\Psi^N(\lambda^-,\lambda^+,\epsilon,K,T)$. So, combining with (2.35) and taking ϵ small shows that (2.26) does hold uniformly on $(N',p,r)\in\bar\Psi_0^N(\lambda^-,\lambda^+,K,T)$, as required for the full statement of Lemma 2.20. □

2.2.4 Proof of Proposition 2.12

Recall that $\mathcal{A}_{N,r} \subseteq \mathcal{F}_N$ is the set of forests on $[N]$ where vertices $1, \dots, r$ are separated. Let F be a uniform choice from $\mathcal{A}_{N,r}$. Then

$$\mathbb{P}(F \in \mathcal{A}_{N,r+1} \,|\, F \in \mathcal{A}_{N,r,k}) = \frac{N-k}{N-r},$$

as the labels of the $k-r$ other vertices in the stack forest containing vertices $[r]$ are uniformly chosen from $\{r+1, \dots, N\}$. Furthermore, $\mathcal{A}_{N,r+1} \subseteq \mathcal{A}_{N,r}$, and so

$$\begin{aligned}
\frac{\mathbb{P}\Big(\bar{G}(N,p) \in \mathcal{A}_{N,r+1}\Big)}{\mathbb{P}\Big(\bar{G}(N,p) \in \mathcal{A}_{N,r}\Big)} &= \mathbb{P}\Big(\bar{G}(N,p) \in \mathcal{A}_{N,r+1} \,|\, \bar{G}(N,p) \in \mathcal{A}_{N,r}\Big) \\
&= \sum_{k=r}^{N} \mathbb{P}\Big(\bar{G}(N,p) \in \mathcal{A}_{N,r+1} \,|\, \bar{G}(N,p) \in \mathcal{A}_{N,r,k}\Big) \\
&\quad \times \mathbb{P}\Big(\bar{G}(N,p) \in \mathcal{A}_{N,r,k} \,|\, \bar{G}(N,p) \in \mathcal{A}_{N,r}\Big) \\
&= \frac{N - \mathbb{E}\Big[k_r(\bar{G}(N,p)) \,|\, \bar{G}(N,p) \in \mathcal{A}_{N,r}\Big]}{N-r}.
\end{aligned}$$

It follows that uniformly on $(N', p, r) \in \bar{\Psi}_0^N(\lambda^-, \lambda^+, K, T)$, as in Lemma 2.20, as $N \to \infty$,

$$N^{1/3}\left[1 - \frac{\mathbb{P}\Big(\bar{G}(N',p) \in \mathcal{A}_{N',r+1}\Big)}{\mathbb{P}\Big(\bar{G}(N',p) \in \mathcal{A}_{N',r}\Big)}\right] - \alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - s\Big) \to 0. \tag{2.36}$$

Now we can return to the increments of $Z^{N,\lambda}$, the exploration process of $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$. Recall Lemma 2.15, which asserts that

$$\mathbb{P}\Big(Z^{N,\lambda}_{m+1} - Z^{N,\lambda}_m = \ell - 1 \,|\, Z^{N,\lambda}_m = r\Big) \propto \mathbb{P}\Big(B^{N-m-r,p} = \ell\Big)$$

$$\times \mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r+\ell-1}\Big), \quad \ell \geq 0,$$

where $B^{N-m-r,p} \sim \mathrm{Bin}(N-m-r,p)$. So we define

$$q_\ell^{N,m,r} := \mathbb{P}\Big(B^{N-m-r,p} = \ell\Big) \times \frac{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r+\ell-1}\Big)}{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r-1}\Big)}. \tag{2.37}$$

Therefore we also have $\mathbb{P}\Big(Z^{N,\lambda}_{m+1} - Z^{N,\lambda}_m = \ell - 1 \,\big|\, Z^{N,\lambda}_m = r\Big) \propto q^{N,m,r}_\ell$. From (2.36), the quotient in (2.37), which we will think of as a *weight*, should be approximately

$$\Big(1 - \alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - \tfrac{m}{N^{2/3}}\Big)N^{-1/3}\Big)^\ell,$$

and so we will be able to approximate $\sum q^{N,m,r}_\ell$ by the probability generating function of $B^{N-m-r,p}$. Indeed, this approximation only breaks down when $r + \ell - 1 \geq KN^{1/3}$, that is, outside the range of (2.36). From now on, we assume $K = 2\rho$. Therefore, for any $\delta > 0$, for large enough N, we have, for all $m \in [0, TN^{2/3}]$, $r \in [1, \rho N^{1/3}]$, and $\ell \leq N^{1/4}$.

$$\frac{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r+\ell-1}\Big)}{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r-1}\Big)} \leq \prod_{i=0}^{\ell-1}\Big(1 - \Big(\alpha\Big(\tfrac{r+i-1}{N^{1/3}}, \lambda - \tfrac{m-1}{N^{2/3}}\Big) - \delta\Big)N^{-1/3}\Big).$$

By uniform continuity of α, since the range of i in this product is asymptotically negligible relative to $N^{1/3}$, we can further say that for large enough N,

$$\frac{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r+\ell-1}\Big)}{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r-1}\Big)} \leq \Big(1 - \Big(\alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - \tfrac{m}{N^{2/3}}\Big) - \delta\Big)N^{-1/3}\Big)^\ell.$$

An identical argument gives

$$\frac{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r+\ell-1}\Big)}{\mathbb{P}\Big(\bar{G}(N-m-1,p) \in \mathcal{A}_{N-m-1,r-1}\Big)} \geq \Big(1 - \Big(\alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - \tfrac{m}{N^{2/3}}\Big) + \delta\Big)N^{-1/3}\Big)^\ell,$$

under the same conditions. From now on, we write $\alpha^N_{m,r} = \alpha\Big(\tfrac{r}{N^{1/3}}, \lambda - \tfrac{m}{N^{2/3}}\Big)$ for brevity.

Keeping $\delta > 0$ fixed, we now address the sums $\sum_{\ell=0}^\infty q^{N,m,r}_\ell$ and $\sum_{\ell=0}^\infty (\ell-1)q^{N,m,r}_\ell$. (Note first that both $q^{N,m,r}_0$ and $q^{N,m,r}_1 \to 1/e$, so these sums are uniformly bounded below.) For large enough N, we have, again for all $m \in [0, TN^{2/3}]$, $r \in [1, \rho N^{1/3}]$,

$$\begin{aligned}\sum_{\ell=0}^{N-m-r} q^{N,m,r}_\ell &\leq \sum_{\ell=0}^{\lceil N^{1/4}\rceil} \mathbb{P}\Big(B^{N-m-r,p} = \ell\Big)\Big(1 - (\alpha^N_{m,r} - \delta)N^{-1/3}\Big)^\ell + \mathbb{P}\Big(B^{N-m-r,p} \geq N^{1/4}\Big)\\ &\leq \Big[(1-p) + p\Big(1 - (\alpha^N_{m,r} - \delta)N^{-1/3}\Big)\Big]^{N-m-r} + \mathbb{P}\Big(B^{N-m-r,p} \geq N^{1/4}\Big)\end{aligned}$$

Now, note that

$$\Big[(1-p)+p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big)\Big]^{N-m-r} = \Big[1-(\alpha^N_{m,r}-\delta)N^{-4/3}+O(N^{-5/3})\Big]^{N-m-r},$$

from which we find that

$$N^{1/3}\bigg[1-\Big[(1-p)+p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big)\Big]^{N-m-r}\bigg]+\Big(\alpha^N_{m,r}-\delta\Big)\to 0, \qquad (2.38)$$

uniformly as $N\to\infty$. The probability $\mathbb{P}\Big(B^{N-m-r,p}\geq N^{1/4}\Big)$ decays exponentially with some positive power of N, so we have shown that for large enough N,

$$\sum_{\ell=0}^{N-m-r} q_\ell^{N,m,r} \leq 1-\Big(\alpha^N_{m,r}-2\delta\Big)N^{-1/3}. \qquad (2.39)$$

Under the same conditions,

$$\sum_{\ell=0}^{N-m-r} q_\ell^{N,m,r} \geq 1-\Big(\alpha^N_{m,r}+2\delta\Big)N^{-1/3}.$$

Now we consider the sum $\sum \ell q_\ell^{N,m,r}$.

$$\begin{aligned}\sum_{\ell=0}^{N-m-r} \ell q_\ell^{N,m,r} &\leq \sum_{\ell=0}^{N-m-r} \ell\mathbb{P}\Big(B^{N-m-r,p}=\ell\Big)\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big)^{\ell} \\ &\qquad + N\mathbb{P}\Big(B^{N-m-r,p}\geq N^{1/4}\Big) \\ &\leq (N-m-r)p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big) \\ &\qquad \times\Big[(1-p)+p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big)\Big]^{N-m-r-1} \\ &\qquad\quad + N\mathbb{P}\Big(B^{N-m-r,p}\geq N^{1/4}\Big).\end{aligned} \qquad (2.40)$$

We can treat the term $\Big[(1-p)+p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big)\Big]^{N-m-r-1}$ as in (2.38). We also have

$$(N-m-r)p\Big(1-(\alpha^N_{m,r}-\delta)N^{-1/3}\Big) = 1+\Big(\lambda-\tfrac{m}{N^{2/3}}-(\alpha^N_{m,r}-\delta)\Big)N^{-1/3}+O(N^{-2/3}). \qquad (2.41)$$

So, in a similar fashion to (2.39), we establish

$$\sum_{\ell=0}^{N-m-r} \ell q_\ell^{N,m,r} \le 1 + \Big(\lambda - 2\alpha_{m,r}^N + 3\delta - \tfrac{m}{N^{2/3}}\Big)N^{-1/3}, \tag{2.42}$$

and

$$\sum_{\ell=0}^{N-m-r} \ell q_\ell^{N,m,r} \ge 1 + \Big(\lambda - 2\alpha_{m,r}^N - 3\delta - \tfrac{m}{N^{2/3}}\Big)N^{-1/3}.$$

Therefore, (where each successive statement holds for large enough N)

$$\begin{aligned}\mathbb{E}\Big[Z_{m+1}^{N,\lambda} - Z_m^{N,\lambda} \,\Big|\, Z_m^{N,\lambda} = r\Big] &= \frac{\sum_{\ell=0}^{N-m-r} \ell q_\ell^{N,m,r} - \sum_{\ell=0}^{N-m-r} q_\ell^{N,m,r}}{\sum_{\ell=0}^{N-m-r} q_\ell^{N,m,r}} \\ &\le \frac{\Big(\lambda - 2\alpha_{m,r}^N + 3\delta - \tfrac{m}{N^{2/3}}\Big)N^{-1/3} + \Big(\alpha_{m,r}^N + 2\delta\Big)N^{-1/3}}{1 + \Big(\lambda - \alpha_{m,r}^N - 2\delta - \tfrac{m}{N^{2/3}}\Big)N^{-1/3}} \\ &\le \Big(\lambda - \alpha_{m,r}^N - \tfrac{m}{N^{2/3}} + 6\delta\Big)N^{-1/3}.\end{aligned}$$

Similarly

$$\mathbb{E}\Big[Z_{m+1}^{N,\lambda} - Z_m^{N,\lambda} \,\Big|\, Z_m^{N,\lambda} = r\Big] \ge \Big(\lambda - \alpha_{m,r}^N - \tfrac{m}{N^{2/3}} - 6\delta\Big)N^{-1/3},$$

and so since $\delta > 0$ was arbitrary, (2.11) follows, completing the proof of Proposition 2.12.

2.2.5 Proof of Proposition 2.13

Variance of increments

We can show (2.12) using the estimates from Section 2.2.4. Recall the definition of $q_\ell^{N,m,r}$ from (2.37). As in (2.40), we have

$$\begin{aligned}\sum_{\ell=0}^{N-m-r} \ell(\ell-1) q_\ell^{N,m,r} \le (N-m-r)(N-m-r-1)p^2\Big(1 - (\alpha_{m,r}^N - \delta)N^{-1/3}\Big)^2 \\ \times \Big[(1-p) + p\Big(1 - (\alpha_{m,r}^N - \delta)N^{-1/3}\Big)\Big]^{N-m-r-2} \\ + N\mathbb{P}\Big(B^{N-m-r,p} \ge N^{1/4}\Big).\end{aligned}$$

Again, we use (2.38) and (2.41). Similarly to (2.42), we have

$$1+\Big(2\lambda-3\alpha^N_{m,r}-4\delta-\tfrac{2m}{N^{2/3}}\Big)N^{-1/3}\le\sum_{\ell=0}^{N-m-r}\ell(\ell-1)q_\ell^{N,m,r}\le 1+\Big(2\lambda-3\alpha^N_{m,r}+4\delta-\tfrac{2m}{N^{2/3}}\Big)N^{-1/3}.$$

In particular, we obtain

$$\sum_{\ell=0}^{N-m-r}(\ell-1)^2q_\ell^{N,m,r}=\sum_{\ell=0}^{N-m-r}\ell(\ell-1)q_\ell^{N,m,r}-\sum_{\ell=0}^{N-m-r}\ell q_\ell^{N,m,r}+\sum_{\ell=0}^{N-m-r}q_\ell^{N,m,r}\to 1,$$

uniformly, which is exactly (2.12).

Jumps in the limit

For any $m\in[N]$,

$$\begin{aligned}\mathbb{P}\Big(|Z^{N,\lambda}_{m+1}-Z^{N,\lambda}_m|>\delta N^{1/3}\Big)&\le\mathbb{P}\Big(\exists v\in[N],\deg_{\bar G(N,p)}(v)>\delta N^{1/3}\Big)\\&\overset{\text{Lemma 2.10}}{\le}\mathbb{P}\Big(\exists v\in[N],\deg_{G(N,p)}(v)>\delta N^{1/3}\Big)\\&\le N\mathbb{P}\Big(\deg_{G(N,p)}(1)>\delta N^{1/3}\Big).\end{aligned}$$

But $\deg_{G(N,p)}(1)\sim\mathrm{Bin}(N-1,p)$, and so for any $\delta>0$, this final term vanishes exponentially fast. So (2.13) follows.

Speed at the boundary

Finally, we check that the discrete processes $(Z^{N,\lambda})$ do not get stuck at zero. Conditional on $Z^{N,\lambda}_m=0$, $Z^{N,\lambda}_{m+1}$, is the number of neighbours of v_{m+1}. Heuristically, we note that if this number of neighbours is either zero or one, there is no possibility for v_{m+1} to be contained in a cycle. By the same argument as led to Lemma 2.15, we have

$$\frac{\mathbb{P}\Big(Z^{N,\lambda}_{m+1}=1\,\big|\,Z^{N,\lambda}_m=0\Big)}{\mathbb{P}\Big(Z^{N,\lambda}_{m+1}=0\,\big|\,Z^{N,\lambda}_m=0\Big)}=\frac{\mathbb{P}\Big(B^{N-m-1,p}=1\Big)}{\mathbb{P}(B^{N-m-1,p}=0)}=\frac{(N-m-1)p}{1-p}.$$

Therefore

$$\liminf_{N\to\infty}\inf_{m\in[0,TN^{2/3}]}\frac{\mathbb{P}\Big(Z^{N,\lambda}_{m+1}=1\,\big|\,Z^{N,\lambda}_m=0\Big)}{\mathbb{P}\Big(Z^{N,\lambda}_{m+1}=0\,\big|\,Z^{N,\lambda}_m=0\Big)}\ge 1,$$

and so

$$\liminf_{N\to\infty} \inf_{m\in[0,TN^{2/3}]} \mathbb{E}\left[\left[Z_{m+1}^{N,\lambda}\right]^2 \mid Z_m^{N,\lambda} = 0\right] \geq \frac{1}{2},$$

as required for (2.14).

This completes the proof of Proposition 2.13.

2.2.6 Regularity of α and proof of Lemma 2.5

In this section, we prove various regularity properties of the function g defined in (2.4), and from this the technical properties we require about α. In particular, the content of Lemma 2.5 is a subset of what follows.

Properties of g

Recall from (2.4) the definition of g

$$g(x) := \frac{1}{\pi}\int_0^\infty \exp(-\tfrac{4}{3}t^{3/2})\cos(xt + \tfrac{4}{3}t^{3/2})\mathrm{d}t.$$

We now prove a lemma which justifies all the regularity properties of g which are required elsewhere.

Lemma 2.22. The function g defined in (2.4) is smooth and positive. Furthermore, it is bounded, uniformly continuous, and satisfies $\int_{x=-\infty}^{\infty} g(x)\mathrm{d}x < \infty$ and $g(x) \to 0$ as $x \to \pm\infty$.

Proof. The key fact, which emerges from Britikov's proof [16] as introduced in Section 2.1.4, is that g is, after stretching by a factor $(2/3)^{2/3}$, the density of the canonical stable distribution with self-similarity exponent $\alpha = 3/2$ and skewness $\beta = -1$. See for example Zolotarev's book [73] for a more general introduction to such distributions and their properties. In particular, g is positive and smooth. Then g is certainly bounded as

$$|g(x)| \leq \frac{1}{\pi}\int_0^\infty \exp\left(-\tfrac{4}{3}t^{3/2}\right)\mathrm{d}t < \infty.$$

It is clear from the Mean Value Theorem that $|\cos(x) - \cos(y)| \leq |x - y|$. Uniform continuity of g then follows as

$$|g(x) - g(y)| \leq \frac{|x-y|}{\pi} \int_0^\infty t \exp\Big(-\tfrac{4}{3}t^{3/2}\Big) \mathrm{d}t,$$

and this integral is finite. Since g is a density, it has finite integral, and then the claim that $g(x) \to 0$ as $x \to \pm\infty$ follows from uniform continuity. □

α is well-defined and monotone

For $k \in \mathbb{N}$, we define

$$J_k(b,\lambda) := \int_0^\infty a^{-k/2} g(\lambda - a) \exp\Big(\tfrac{(\lambda-a)^3}{6}\Big) \exp\Big(-\tfrac{b^2}{2a}\Big) \mathrm{d}a, \quad b > 0, \lambda \in \mathbb{R}. \tag{2.43}$$

Lemma 2.23. For each $k \in \mathbb{N}$, this function J_k is well-defined and continuous, and has partial derivative with respect to b given by

$$\frac{\partial}{\partial b} J_k(b,\lambda) = -b J_{k+2}(b,\lambda). \tag{2.44}$$

Furthermore, the function $\alpha(b,\lambda) := \frac{J_1(b,\lambda)}{J_3(b,\lambda)}$ defined in (2.5) is also well-defined, continuous and differentiable with respect to b.

Proof. To show that $J_k(b,\lambda) < \infty$, we consider the integral in (2.43) separately over the ranges $a \in (0,1]$ and $a \in [1,\infty)$. We have

$$\int_1^\infty a^{-k/2} g(\lambda - a) \exp\Big(\tfrac{(\lambda-a)^3}{6}\Big) \exp\Big(-\tfrac{b^2}{2a}\Big) \mathrm{d}a < e^{\lambda^3/6} \int_1^\infty g(\lambda - a) \mathrm{d}a < \infty, \tag{2.45}$$

and

$$\int_0^1 a^{-k/2} g(\lambda - a) \exp\Big(\tfrac{(\lambda-a)^3}{6}\Big) \exp\Big(-\tfrac{b^2}{2a}\Big) \mathrm{d}a < e^{\lambda^3/6} g_{\max} \int_0^1 a^{-k/2} \exp\Big(-\tfrac{b^2}{2a}\Big) \mathrm{d}a < \infty. \tag{2.46}$$

Thus we have $J_k(b,\lambda) < \infty$.

Since the bounds (2.45) and (2.46) hold locally uniformly in (b, λ), continuity of J_k follows from the dominated convergence theorem.

We introduce some notation for the integrand in (2.43):

$$j_k(a, b, \lambda) := a^{-k/2} g(\lambda - a) \exp\Big(\frac{(\lambda-a)^3}{6}\Big) \exp\Big(-\frac{b^2}{2a}\Big).$$

Fix $b_0 > 0$, and let $B \subseteq \mathbb{R}_+$ be some interval containing b_0. Clearly both $j_k(a, b, \lambda)$ and $\frac{\partial}{\partial b} j_k(a, b, \lambda)$ are continuous on $\mathbb{R}_+ \times B \times \mathbb{R}$. Now take some $b^- < \inf B$ and $b^+ > \sup B$. Then we have

$$j_k(a, b, \lambda) \leq j_k(a, b^-, \lambda), \quad \Big|\frac{\partial}{\partial b} j_k(a, b, \lambda)\Big| \leq b^+ j_{k+2}(a, b^-, \lambda), \quad \forall b \in B, a > 0, \lambda \in \mathbb{R}.$$

We have already shown that $\int_0^\infty j_k(a, b^-, \lambda) \mathrm{d}a < \infty$ and $\int_0^\infty j_{k+2}(a, b^-, \lambda) \mathrm{d}a < \infty$. So we may differentiate (2.43) inside the integral at $b = b_0$ (and b_0 was arbitrary), to obtain, for all $k \geq 1$,

$$\frac{\partial}{\partial b} J_k(b, \lambda) = -b J_{k+2}(b, \lambda). \tag{2.47}$$

Well-definedness and continuity of $\alpha(b, \lambda) := \frac{J_1(b,\lambda)}{J_3(b,\lambda)}$ follow immediately, since $J_3(b, \lambda) > 0$ for all $b > 0, \lambda \in \mathbb{R}$, and furthermore $\alpha(b, \lambda)$ is differentiable in its first argument as required, with

$$\frac{\partial}{\partial b} \alpha(b, \lambda) = \frac{b J_1(b, \lambda) J_5(b, \lambda)}{J_3(b, \lambda)^2} - b, \tag{2.48}$$

through two applications of (2.47). □

Proposition 2.24. $\alpha(b, \lambda)$ is increasing as a function of b.

Proof. Heuristically, we can view (2.5) as the expectation of a with respect to the measure with density $a^{-3/2} \exp\Big(\frac{(\lambda-a)^3}{6}\Big) g(\lambda - a)$, weighted by a factor $\exp(-\frac{b^2}{2a})$. Increasing b reweights in favour of larger values of a, so $\alpha(b, \lambda)$ is increasing in b. We make this formal with the following lemma.

Lemma 2.25. Let f, h be functions $\mathbb{R}_+ \to \mathbb{R}_+$ such that h is strictly increasing, and the integrals

$$\int_0^\infty af(a)h(a)\mathrm{d}a, \quad \int_0^\infty f(a)\mathrm{d}a, \tag{2.49}$$

exist and are finite. Then

$$\frac{\int_0^\infty af(a)h(a)\mathrm{d}a}{\int_0^\infty f(a)h(a)\mathrm{d}a} > \frac{\int_0^\infty af(a)\mathrm{d}a}{\int_0^\infty f(a)\mathrm{d}a}. \tag{2.50}$$

Proof. Note if the integrals in (2.49) exist, then so do the remaining two integrals that appear in (2.50). First, it is clear that for any $c > 0$

$$\int_{a=c}^\infty \int_{b=0}^c f(a)f(b)h(a)\mathrm{d}b\mathrm{d}a > \int_{a=c}^\infty \int_{b=0}^c f(a)f(b)h(b)\mathrm{d}b\mathrm{d}a,$$

since $h(a) > h(b)$ over this range. But clearly also we have

$$\int_{a=c}^\infty \int_{b=c}^\infty f(a)f(b)h(a)\ \mathrm{d}b\ \mathrm{d}a = \int_{a=c}^\infty \int_{b=c}^\infty f(a)f(b)h(b)\ \mathrm{d}b\ \mathrm{d}a.$$

Adding these two relations, and integrating over $c \in \mathbb{R}_+$, we obtain

$$\int_{c=0}^\infty \int_{a=c}^\infty \int_{b=0}^\infty f(a)f(b)h(a)\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}c > \int_{c=0}^\infty \int_{a=c}^\infty \int_{b=0}^\infty f(a)f(b)h(b)\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}c$$
$$\int_{a=0}^\infty \int_{b=0}^\infty \int_{c=0}^a f(a)f(b)h(a)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a > \int_{a=0}^\infty \int_{b=0}^\infty \int_{c=0}^a f(a)f(b)h(b)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a$$
$$\left(\int_{a=0}^\infty af(a)h(a)\,\mathrm{d}a\right)\left(\int_{b=0}^\infty f(b)\,\mathrm{d}b\right) > \left(\int_{a=0}^\infty af(a)\,\mathrm{d}a\right)\left(\int_{b=0}^\infty f(b)h(b)\,\mathrm{d}b\right),$$

exactly as required. □

To apply the lemma to α, fix $\lambda \in \mathbb{R}$ and let $b' > b$, and set

$$f(a) := a^{-3/2}g(\lambda - a)\exp\left(\tfrac{(\lambda-a)^3}{6}\right)\exp\left(-\tfrac{b^2}{2a}\right),$$

and

$$h(a) := \exp\left(-\tfrac{b'^2-b^2}{2a}\right),$$

which is strictly increasing. We conclude from the lemma that

$$\alpha(b', \lambda) > \alpha(b, \lambda), \quad b' > b.$$

This completes the proof of Proposition 2.24. □

Lipschitz property of α

The following proposition establishes the behaviour of $\alpha(b, \lambda)$ as $b \downarrow 0$ in the sense required to complete the proof of Lemma 2.20. It also establishes a Lipschitz condition for α, required in Proposition 2.6 for the well-posedness of the reflected SDE (2.6).

Proposition 2.26. Given $-\infty < \lambda^- < \lambda^+ < \infty$, we have

$$\lim_{b \downarrow 0} \sup_{\lambda \in [\lambda^-, \lambda^+]} \alpha(b, \lambda) = 0. \tag{2.51}$$

Furthermore, given $\rho < \infty$, there exists a constant $C < \infty$ such that α satisfies the Lipschitz condition

$$\left|\alpha(b, \lambda) - \alpha(b', \lambda)\right| \leq C|b - b'|, \quad b, b' \in (0, \rho], \, \lambda \in [\lambda^-, \lambda^+]. \tag{2.52}$$

Proof. To show (2.52), it suffices to prove the following:

$$\sup_{b \in (0, \rho], \lambda \in [\lambda^-, \lambda^+]} \left| \frac{\partial}{\partial b} \alpha(b, \lambda) \right| < \infty. \tag{2.53}$$

The steps we take to prove (2.52) will also allow us to read off (2.51). Recall the expression (2.48) from the proof of Lemma 2.23:

$$\frac{\partial}{\partial b} \alpha(b, \lambda) = \frac{b J_1(b, \lambda) J_5(b, \lambda)}{J_3(b, \lambda)^2} - b. \tag{2.48}$$

From Lemma 2.23, we know that $\frac{\partial}{\partial b}\alpha(b, \lambda)$ is continuous, and so to verify (2.53), it remains to consider the limit as $b \downarrow 0$. We examine the behaviour of each of $J_1(b, \lambda), J_3(b, \lambda), J_5(b, \lambda)$ in this limit.

First, we consider J_1. We define

$$\gamma_1(\lambda) := e^{\lambda^3/6} \int_0^\infty a^{-1/2} g(\lambda - a) \mathrm{d}a,$$

which is seen to be finite by a similar decomposition to (2.45) and (2.46). Then

$$\begin{aligned}\gamma_1(\lambda) - J_1(b,\lambda) &\leq g_{\max} \int_0^\infty a^{-1/2} \Big[e^{\lambda^3/6} - \exp\Big(\tfrac{(\lambda-a)^3}{6}\Big) \exp\Big(-\tfrac{b^2}{2a}\Big)\Big] \mathrm{d}a \\ &\leq e^{\lambda^3/6} g_{\max} \int_0^\infty a^{-1/2} \Big[1 - \exp\Big(-\tfrac{b^2}{2a}\Big)\Big] \mathrm{d}a,\end{aligned}$$

and so by monotone convergence we have as $b \downarrow 0$,

$$\sup_{\lambda \in (-\infty, \lambda^+]} |J_1(b,\lambda) - \gamma_1(\lambda)| \to 0. \tag{2.54}$$

Substituting $u = \frac{b^2}{2a}$ into (2.43) gives

$$J_3(b,\lambda) = \frac{\sqrt{2}}{b} \int_0^\infty u^{-1/2} g\Big(\lambda - \tfrac{b^2}{2u}\Big) \exp\left(\frac{(\lambda - \frac{b^2}{2u})^3}{6}\right) \exp(-u) \mathrm{d}u.$$

So we define

$$\gamma_3(\lambda) := \sqrt{2} g(\lambda) e^{\lambda^3/6} \int_0^\infty u^{-1/2} \exp(-u) \mathrm{d}u,$$

and then by dominated convergence and uniform continuity of g,

$$\lim_{b \downarrow 0} \sup_{\lambda \in (-\infty, \lambda^+]} |bJ_3(b,\lambda) - \gamma_3(\lambda)| = 0. \tag{2.55}$$

A very similar argument can be deployed to obtain

$$\lim_{b \downarrow 0} \sup_{\lambda \in (-\infty, \lambda^+]} \Big| b^3 J_5(b,\lambda) - \gamma_5(\lambda) \Big| = 0,$$

where

$$\gamma_5(\lambda) := 2\sqrt{2} g(\lambda) e^{\lambda^3/6} \int_0^\infty u^{1/2} \exp(-u) \mathrm{d}u.$$

So we can return to (2.48), which we rewrite as

$$\frac{\partial}{\partial b}\alpha(b,\lambda) = \frac{J_1(b,\lambda)\cdot b^3 J_5(b,\lambda)}{(bJ_3(b,\lambda))^2} - b.$$

We now take the limit $b \downarrow 0$, for $\lambda \in [\lambda^-, \lambda^+]$. The denominator is uniformly bounded away from zero for $\lambda \in [\lambda^-, \lambda^+]$. So we obtain

$$\lim_{b\downarrow 0} \sup_{\lambda\in[\lambda^-,\lambda^+]} \left| \frac{\partial}{\partial b}\alpha(b,\lambda) - \frac{\gamma_1(\lambda)\gamma_5(\lambda)}{\gamma_3(\lambda)^2} \right| = 0. \tag{2.56}$$

Now, γ_3, γ_5 are clearly continuous, and γ_1 is also continuous by the same argument as given for continuity of J_k in the proof of Lemma 2.23. Furthermore, γ_3 is positive, and so we have

$$\max_{\lambda\in[\lambda^-,\lambda^+]} \gamma_1(\lambda) < \infty, \qquad \min_{\lambda\in[\lambda^-,\lambda^+]} \gamma_3(\lambda) > 0.$$

Taken with (2.55), the latter shows that

$$\lim_{b\downarrow 0} \inf_{\lambda\in[\lambda^-,\lambda^+]} J_3(b,\lambda) = \infty.$$

Therefore, since $\alpha(b,\lambda) = \frac{J_1(b,\lambda)}{J_3(b,\lambda)}$, using (2.54) as well, we obtain precisely the first required statement (2.51).

For similar reasons, we have

$$\max_{\lambda\in[\lambda^-,\lambda^+]} \frac{\gamma_1(\lambda)\gamma_5(\lambda)}{\gamma_3(\lambda)^2} < \infty. \tag{2.57}$$

Since $\frac{\partial}{\partial b}\alpha(b,\lambda)$ is continuous on $(0,\rho] \times [\lambda^-, \lambda^+]$, from (2.56) and (2.57), it's clear that

$$\sup_{b\in(0,\rho],\lambda\in[\lambda^-,\lambda^+]} \left| \frac{\partial}{\partial b}\alpha(b,\lambda) \right| < \infty,$$

from which (2.52) follows. This completes the proof of Proposition 2.26. □

2.2.7 Existence of Z^λ and proof of Theorem 2.8

First we prove Proposition 2.6, which asserts that Z^λ is well-defined.

THEOREM 2.27. [62, §IX 2.14] Let $\sigma(s,x)$ and $b(s,x)$ be functions $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, and W a Brownian motion. For $z_0 \geq 0$, we call a solution to the SDE with reflection $e_{z_0}(\sigma, b)$ a pair (Z, K) of processes such that

1. the process Z is continuous, non-negative, $\mathcal{F}^W$-adapted, and

$$Z(t) = z_0 + \int_0^t \sigma(s, Z(s))\mathrm{d}W(s) + \int_0^t b(s, Z(s))\mathrm{d}s + K(t), \tag{2.58}$$

2. the process K is continuous, non-decreasing, vanishing at zero, $\mathcal{F}^W$-adapted, and

$$\int_0^\infty Z(s)\mathrm{d}K(s) = 0. \tag{2.59}$$

If σ and b are bounded and satisfy the global Lipschitz condition

$$|\sigma(s,x) - \sigma(s,y)| + |b(s,x) - b(s,y)| \leq C|x - y|,$$

for every $s, x, y \in (0, \infty)$ and some constant C, then there exists a solution to $e_{z_0}(\sigma, b)$, and furthermore this solution is unique.

Proof of Proposition 2.6

We now return to the existence of Z^λ as in (2.6), for fixed $\lambda \in \mathbb{R}$. In this setting $\sigma(s,x) \equiv 1$, but

$$b(s,x) := \lambda - s - \alpha(x, \lambda - s), \tag{2.60}$$

is neither bounded below nor satisfies the global Lipschitz property. However, by Proposition 2.26, for any $R > 0$, we can define $b^R(s,x)$ such that $b^R(s,x)$ is bounded and globally Lipschitz in x; and $b^R(s,x) = b(s,x)$ whenever $(s,x) \in [0,R] \times [0,R]$. Then Theorem 2.27 asserts that there is a unique pair of processes $(Z^{\lambda,R}, K^{\lambda,R})$ corresponding to this drift, where $Z^{\lambda,R}(0) = 0$.

Let $\tau^{\lambda,R}$ be the time at which $Z^{\lambda,R}$ first hits R. Take $R' \geq R$. Then, it is clear that $Z^{\lambda,R}$ is equal to $Z^{\lambda,R'}$ up to time $R \wedge \tau^{\lambda,R}$ almost surely. Also, since $b(s,x)$ is bounded above by λ, it follows that $\tau^{\lambda,R} \to \infty$ as $R \to \infty$ almost surely. Therefore, we may define

$$Z^{\lambda}(t) = \lim_{R\to\infty} Z^{\lambda,R}(t),$$

for almost all paths of W, and Z^{λ}. It is immediate that Z^{λ} satisfies (2.6). Furthermore, any solution $(Z^{\lambda}, K^{\lambda})$ to (2.6) must coincide with $(Z^{\lambda,R}, K^{\lambda,R})$ up to $\tau^{\lambda,R}$, and so uniqueness of $(Z^{\lambda}, K^{\lambda})$ follows as well, as required for Proposition 2.6.

Convergence of non-negative Markov processes

A general framework for showing convergence of Markov processes to the solutions of SDEs was introduced by Stroock and Varadhan in the 60s (see, for example, [67]). The convergence of Markov processes to reflected diffusions is treated in [66] in high generality, allowing for general boundaries in $\mathbb{R}^d$, and inhomogeneous stickiness at the boundaries.

We summarise the conditions and statement of Theorem 6.3 from [66] in a relevant special case:

THEOREM 2.28. Suppose a family of non-negative-real-valued Markov processes $(Z^N_m)_{m\geq 0}$ is given, and a pair of functions $\sigma(t,x)$ and $b(t,x)$ satisfying the conditions of Theorem 2.27. Now, fix $T > 0$, and suppose that $Z^N_0 = 0$, and the following hold

$$N^{1/3}\mathbb{E}\Big[Z^N_{tN^{2/3}+1} - Z^N_{tN^{2/3}} \,\big|\, Z^N_{tN^{2/3}} = xN^{1/3}\Big] \to b(t,x),$$

$$\mathbb{E}\Big[\Big[Z^N_{tN^{2/3}+1} - Z^N_{tN^{2/3}}\Big]^2 \,\big|\, Z^N_{tN^{2/3}} = xN^{1/3}\Big] \to \sigma(t,x)^2,$$

$$N^{2/3}\mathbb{P}\Big(Z^N_{tN^{2/3}+1} - Z^N_{tN^{2/3}} > \delta \,\big|\, Z^N_{tN^{2/3}} = xN^{1/3}\Big) \to 0,$$

as $N \to \infty$, for any $\delta > 0$, uniformly for $t \in [0,T]$ and x in any compact interval in $(0,\infty)$. Finally assume that for some $\epsilon > 0$,

$$\liminf_{N\to\infty} \inf_{\substack{x\in N^{-1/3}\mathbb{Z}\cap[0,\epsilon] \\ t\in[0,T]}} \mathbb{E}\Big[\Big[Z^N_{tN^{2/3}+1} - Z^N_{tN^{2/3}}\Big]^2 \mid Z^N_{tN^{2/3}} = xN^{1/3}\Big] > 0. \tag{2.61}$$

Then, defining

$$\tilde{Z}^N(t) := N^{-1/3} Z^N_{\lfloor tN^{2/3} \rfloor}, \quad t \in [0,T], \tag{2.62}$$

we have $\tilde{Z}^N \xrightarrow{d} Z$, uniformly on $[0,T]$, where Z is the unique solution to the reflected SDE $e_0(\sigma, b)$, as defined in Theorem 2.27.

Note. The condition (2.61) is required to ensure instantaneous reflection at the boundary, rather than absorption, or sticky reflection.

Proof of Theorem 2.8

Now let $Z^{N,\lambda}$ be the exploration process of $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$. Again, in our setting, we must account for the fact that the drift of Z^λ is neither bounded nor globally Lipschitz.

Recall from (2.60) and the following paragraph the definitions of $b(s,x)$ and $b^R(s,x)$. For any $R \in \mathbb{N}$, we can construct a Markov process $(Z^{N,\lambda,R}_m, m \geq 0)$ whose transition probabilities coincide with those of $Z^{N,\lambda}$ whenever $m \in [0, TN^{2/3}]$ and $Z^{N,\lambda,R}_m \leq RN^{1/3}$, and for which, by Proposition 2.12,

$$N^{1/3}\mathbb{E}\Big[Z^{N,\lambda,R}_{tN^{2/3}+1} - Z^{N,\lambda,R}_{tN^{2/3}} \mid Z^{N,\lambda,R}_{tN^{2/3}} = xN^{1/3}\Big] \to b^R(t,x),$$

uniformly for $t \in [0,T]$ and x in any compact interval in $(0,\infty)$. Thus $\tilde{Z}^{N,\lambda,R}$, defined from $Z^{N,\lambda,R}$ as in (2.62) satisfies $\tilde{Z}^{N,\lambda,R} \xrightarrow{d} Z^{\lambda,R}$ uniformly on $[0,T]$.

From this, it is clear that

$$\mathbb{P}\left(\sup_{m\in[0,TN^{2/3}]} Z^{N,\lambda,R} > RN^{1/3} \right) \to 0,$$

as $R \to \infty$, and so as processes on $[0,T]$, the law of $\tilde{Z}^{N,\lambda,R}$ converges to the law of $\tilde{Z}^{N,\lambda}$ as $R \to \infty$, and the law of $Z^{\lambda,R}$ converges to the law of Z^{λ}. Thus we obtain (2.8).

2.3 Excursions and component sizes

In this section, we will prove that Theorem 2.7 follows from Theorem 2.8.

As in Aldous [5], we must check that excursions of the limiting reflected SDE are matched by excursions of the discrete exploration processes. In particular, it must happen with vanishing probability that a zero of the limiting process appears only as the limit of small *positive* local minima of the discrete processes. In addition, we must show that there are with high probability no large discrete components which appear late enough in the exploration that they are not represented in the limit. Several stages of the argument will be based on a comparison of $\bar{G}(N,p)$ and the original model $G(N,p)$, for which some of the results are easier, or known.

2.3.1 Sizes and labels of critical components

First, we establish the notation we will use to describe the sequence of rescaled component sizes. Fix $T > 0$, then:

- Let $(C_1^{N,\lambda}, C_2^{N,\lambda}, \dots)$ be the sequence of sizes of components of $\bar{G}\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$, in non-increasing order.
- Analogously, let $(C_1^{N,\lambda,T}, C_2^{N,\lambda,T}, \dots)$ be the sequence of sizes of components of $\bar{G}\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$ which have non-empty intersection with $\{v_1, \dots, v_{\lfloor TN^{2/3} \rfloor}\}$, an initial segment of the breadth-first ordering introduced in Section 2.1.3. That is, least one vertex has been seen by step $\lfloor TN^{2/3} \rfloor$ of the exploration process. Again, we assume the sequence is ordered such that $C_1^{N,\lambda,T} \ge C_2^{N,\lambda,T} \ge \dots$.

Lemma 2.29. Fix $\lambda^+ \in \mathbb{R}$. Then

$$\lim_{\gamma \to \infty} \limsup_{N \to \infty} \sup_{\lambda \le \lambda^+} \mathbb{P}\Big(C_1^{N,\lambda} \ge \gamma N^{2/3}\Big) = 0. \tag{2.63}$$

Proof. By Lemma 2.10, it suffices to show (2.63) when $C_1^{N,\lambda}$ is the size of the largest component in the *unconditioned* random graph $G\Big(N, \frac{1+\lambda N^{-1/3}}{N}\Big)$. Let $C^{N,\lambda}(v)$ be the component of $G(N, \frac{1+\lambda N^{-1/3}}{N})$ containing v, a uniformly-chosen vertex in $[N]$. Observe that for $\lambda \le \lambda^+$,

$$\mathbb{E}\Big[|C^{N,\lambda^+}(v)|\Big] \ge \mathbb{E}\Big[|C^{N,\lambda}(v)|\Big] \ge \frac{\gamma N^{2/3}}{N} \cdot \gamma N^{2/3} \mathbb{P}\Big(C_1^{N,\lambda} \ge \gamma N^{2/3}\Big).$$

Therefore, from Proposition 2.11,

$$\limsup_{N\to\infty} \sup_{\lambda \le \lambda^+} \mathbb{P}\Big(C_1^{N,\lambda} \ge \gamma N^{2/3}\Big) \le \frac{\Theta^{\lambda^+}}{\gamma^2},$$

and so (2.63) follows. □

The following lemma shows that critical components will with high probability include a vertex with label $O(N^{1/3})$.

Lemma 2.30. Fix $\epsilon > 0$, and $\lambda^+ \in \mathbb{R}$. Then

$$\lim_{\Gamma\to\infty} \limsup_{N\to\infty} \sup_{\lambda \le \lambda^+} \mathbb{P}\Big(\exists \text{ cpt } C \text{ in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big) \,:\, |C| \ge \epsilon N^{2/3} \quad \text{and } C \cap \Big\{1, \dots, \lfloor \Gamma N^{1/3} \rfloor\Big\} = \varnothing\Big) = 0. \tag{2.64}$$

Proof. Applying Markov's inequality to (2.10), and summing over all vertices,

$$\limsup_{N\to\infty} \sup_{\lambda \le \lambda^+} N^{-2/3} \mathbb{E}\Big[\Big|\Big\{v \in [N] \,:\, \Big|C^{N,\lambda}(v)\Big| \ge \epsilon N^{2/3}\Big\}\Big|\Big] \le \frac{\Theta^{\lambda^+}}{\epsilon}.$$

Therefore,

$$\limsup_{N\to\infty} \sup_{\lambda \le \lambda^+} \mathbb{E}\Big[\#\text{cpts } C \text{ in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big) \text{ s.t. } |C| \ge \epsilon N^{2/3}\Big] \le \frac{\Theta^{\lambda^+}}{\epsilon^2}.$$

Then, since the labelling is independent of the component sizes in $\bar{G}(N, p)$,

$$\limsup_{N\to\infty} \sup_{\lambda \le \lambda^+} \mathbb{E}\Big[\#\text{cpts } C \text{ in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big) \text{ s.t. } |C| \ge \epsilon N^{2/3} \quad \text{and } C \cap \Big\{1, \dots, \lfloor \Gamma n^{1/3} \rfloor\Big\} = \varnothing\Big]$$

$$\leq \frac{\Theta^{\lambda^+}}{\epsilon^2} \times \lim_{N\to\infty} \frac{\binom{N-\lfloor \Gamma N^{1/3}\rfloor}{\lfloor \epsilon N^{2/3}\rfloor}}{\binom{N}{\lfloor \epsilon N^{2/3}\rfloor}}$$
$$\leq \frac{\Theta^{\lambda^+}}{\epsilon^2} \exp(-\Gamma\epsilon),$$

where the final deduction follows by a routine application of Stirling's approximation. The result follows immediately. □

We now use the previous result to show that the largest components will typically appear near the start of the exploration process. This will be important later, since if large critical components appear arbitrarily late in the exploration process, then they cannot be treated via convergence on compact intervals.

Lemma 2.31. Fix $\epsilon > 0$ and $\lambda^+ \in \mathbb{R}$ as before. Then

$$\lim_{T\to\infty} \limsup_{N\to\infty} \sup_{\lambda\leq\lambda^+} \mathbb{P}\Big(\exists \text{ cpt } C \text{ in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big) : |C| \geq \epsilon N^{2/3} \quad \text{and } C \cap \Big\{v_1, \dots, v_{\lfloor TN^{2/3}\rfloor}\Big\} = \varnothing\Big) = 0. \tag{2.65}$$

Proof. Fix $\Gamma > 0$. We define the events

$$A^{N,\Gamma,T} := \{|C^N(1)| + |C^N(2)| + \ldots + |C^N(\lfloor \Gamma N^{1/3}\rfloor)| > TN^{2/3}\},$$

$$B^{N,\epsilon,\Gamma} := \Big\{\exists \text{ cpt } C \text{ in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big) : |C| \geq \epsilon N^{2/3},\, C \cap \Big\{1, \dots, \lfloor \Gamma N^{1/3}\rfloor\Big\} = \varnothing\Big\},$$

as in Lemma 2.30. Then, by Markov's inequality,

$$\mathbb{P}\Big(A^{N,\Gamma,T} \text{ holds in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big)\Big) \leq \frac{\Gamma N^{1/3}\mathbb{E}\Big[|C^{N,\lambda}(1)|\Big]}{TN^{2/3}},$$

So by Proposition 2.11

$$\limsup_{N\to\infty} \sup_{\lambda\leq\lambda^+} \mathbb{P}\Big(A^{N,\Gamma,T} \text{ holds in } \bar{G}\Big(N, \tfrac{1+\lambda N^{-1/3}}{N}\Big)\Big) \leq \frac{\Theta^{\lambda^+}\Gamma}{T}. \tag{2.66}$$

Whenever $\bar{G}(N, \frac{1+\lambda N^{-1/3}}{N})$ contains a component of size at least $\epsilon N^{2/3}$ which is not exhausted during the first $TN^{2/3}$ steps of the exploration process, at least one of $A^{N,\Gamma,T}$

and $B^{N,\epsilon,\Gamma}$ must hold. So take $\Gamma = \sqrt{T}$, then let $T \to \infty$. By (2.66) and Lemma 2.30, the result follows. □

2.3.2 Components and excursions up to time T - notation and outline

We first show that for any $T < \infty$ the excursion lengths in the exploration processes on the interval $[0, T]$ appear correctly in the limit.

In everything that follows, we work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose existence is guaranteed by the Skorohod representation theorem, where $\tilde{Z}^{N,\lambda} \overset{\mathbb{P}-\text{a.s.}}{\to} Z^\lambda$ with respect to the topology of uniform convergence on compact intervals. Recall that throughout this section, $\lambda \in \mathbb{R}$ is fixed, and $p = \frac{1+\lambda N^{-1/3}}{N}$, so henceforth we suppress λ from the notation.

Let $C_1^T \geq C_2^T \geq \dots$ be the lengths of excursions of $\tilde{Z}^\lambda$ above zero which have non-empty intersection with $[0, T]$, in non-increasing order. We will prove the following convergence result for the components seen within the first $TN^{2/3}$ steps of the exploration process.

Proposition 2.32. Fix $T > 0$ and $k \geq 1$. Then as $N \to \infty$,

$$N^{-2/3}(C_1^{N,T}, C_2^{N,T}, \dots, C_k^{N,T}) \quad \overset{d}{\to} \quad (C_1^T, C_2^T, \dots, C_k^T). \tag{2.67}$$

The proof occupies the rest of this subsection. Throughout, we write

$$C^{N,T} = (C_1^{N,T}, \dots, C_k^{N,T}), \quad \text{and} \quad C^T = (C_1^T, \dots, C_k^T).$$

The concern is that the reflected exploration process might regularly approach zero without actually hitting zero, and thus starting a new component. To show that this effect does not appear in the limit, we use the fact that the components of $\bar{G}(N, p)$ have the structure of uniform random trees. Then we can approximate the exploration process within a component by a Brownian excursion, and show that the probability of zeros in the limit which do not correspond to the start or end of a component is small.

Definition 2.33. Given two sequences $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_k)$, let $a^\downarrow, b^\downarrow$ denote the sequences rearranged into non-increasing order. Then, we say $a \succeq b$ or a *weakly majorises* b if for every $\ell \leq k$,

$$\sum_{i=1}^{\ell} a_i^\downarrow \geq \sum_{i=1}^{\ell} b_i^\downarrow.$$

It is easy to check that this gives a pre-order on $(\mathbb{R} \cup \{\infty\})^k$, and a partial order on non-increasing sequences finer than the standard ordering.

We will prove Proposition 2.32 by stochastically sandwiching C^T between any weak limit of $C^{N,T}$, and any weak limit of a related sequence of lengths $C^{N,T,\delta}$ associated with $\tilde{Z}^N$, which will be defined shortly. This stochastic ordering will be with respect to weak majorisation. The two directions of this sandwiching argument occupy the next two sections. Finally, we show that for small enough δ, these outer distributions are close in the sense of the Lévy–Prohorov metric.

2.3.3 Limits of component sizes stochastically majorise excursion lengths

We show that limit points of $C^{N,T}$ majorise C^T, $\mathbb{P}$-almost surely.

For any reference time $s \in [0, T]$, we define

$$\alpha(s) := \sup\{t \leq s : Z(t) = 0\}, \quad \alpha^N(s) := \sup\{t \leq s : \tilde{Z}^N(t) = 0\},$$

$$\beta(s) := \inf\{t \in [s, \infty) : Z(t) = 0\}, \quad \beta^N(s) := \inf\{t \in [s, T] : \tilde{Z}^N(t) = 0\}.$$

It will be convenient to avoid values of s where α^N and β^N are non-constant, so we define

$$\bar{\mathbb{Q}} := \bigcup_{N \in \mathbb{N}} N^{-2/3}\mathbb{Z}.$$

We also define the event

$$\Psi^T := \Big\{ \tilde{Z}^N \to Z \text{ uniformly on } [0, \beta(T)],\ Z \text{ continuous on } [0, \beta(T)] \Big\}.$$

Since $\beta(T)<\infty$ almost surely, and $\tilde{Z}^N \to Z$ uniformly on compact intervals, we have $\mathbb{P}\Big(\Psi^T\Big)=1$. It follows easily that on Ψ^T,

$$\limsup_{N\to\infty}\alpha^N(s)\le\alpha(s),\quad \liminf_{N\to\infty}\beta^N(s)\ge\beta(s),\quad \forall s\in[0,T]. \tag{2.68}$$

Now, on Ψ^T, given Z, choose $s_1,\dots,s_k\in[0,T]\backslash\bar{\mathbb{Q}}$ such that each s_i lies in the ith longest excursion of Z, which has non-empty intersection with $[0,T]$. That is, $\beta(s_i)-\alpha(s_i)=C_i^T$. Now consider any limit point

$$(\bar\alpha(s_1),\dots,\bar\alpha(s_k),\bar\beta(s_1),\dots,\bar\beta(s_k),\bar C_1^T,\dots,\bar C_k^T), \tag{2.69}$$

of $(\alpha^N(s_1),\dots,\alpha^N(s_k),\beta^N(s_1),\dots,\beta^N(s_k),C_1^{N,T},\dots,C_k^{N,T})$, as $N\to\infty$, where we allow $\bar C_1^T$ and at most one of the $\bar\beta(s_i)$ to be ∞. By compactness, we can be sure that there are such limit points. To avoid introducing extra notation, we will assume that (2.69) is a true limit, rather than a subsequential limit.

By (2.68), for any $\ell\le k$,

$$\bigcup_{i=1}^{\ell}[\bar\alpha(s_i),\bar\beta(s_i)]\supseteq\bigcup_{i=1}^{\ell}[\alpha(s_i),\beta(s_i)],$$

where the sets in the union on the right-hand side have disjoint interiors. By construction of $\alpha^N(s_i),\beta^N(s_j)$, any pair of intervals $[\alpha^N(s_i),\beta^N(s_i)]$ and $[\alpha^N(s_j),\beta^N(s_j)]$ are either equal or disjoint. Therefore the intervals in the union on the left-hand side are either equal or have disjoint interiors. So let $\Gamma_\ell\subseteq[\ell]$ be some set of indices such that

$$[\bar\alpha(s_i),\bar\beta(s_i)]\ne[\bar\alpha(s_j),\bar\beta(s_j)],\quad \forall i\ne j\in\Gamma_\ell,\quad\text{and}\quad \bigcup_{i\in\Gamma_\ell}[\bar\alpha(s_i),\bar\beta(s_i)]\supseteq\bigcup_{i=1}^{\ell}[\alpha(s_i),\beta(s_i)].$$

Furthermore, we may demand $\Gamma_1\subseteq\Gamma_2\subseteq\ldots\subseteq\Gamma_k$. Thus

$$\sum_{i\in\Gamma_\ell}(\bar\beta(s_i)-\bar\alpha(s_i))\ge\sum_{i=1}^{\ell}(\beta(s_i)-\alpha(s_i)).$$

That is,

$$\left(\bar{\beta}(s_1)-\bar{\alpha}(s_1),\ldots,\bar{\beta}\left(s_{|\Gamma_k|}\right)-\bar{\alpha}\left(s_{|\Gamma_k|}\right),0,\ldots,0\right)\succeq(\beta(s_1)-\alpha(s_1),\ldots,\beta(s_k)-\alpha(s_k)). \tag{2.70}$$

For any N, and any $s\in[0,T]\backslash\bar{\mathbb{Q}}$, the interval $[\alpha^N(s),\beta^N(s)]$ is associated via the reflected exploration process with exactly one component of $\bar{G}(N,p)$. The size of this component is at least $(\beta^N(s)-\alpha^N(s))N^{2/3}$.

Note. The two cases where the size of the component is not equal to $(\beta^N(s)-\alpha^N(s))N^{2/3}$ are: 1) when $\beta^N(s)=\infty$; 2) when $\tilde{Z}^N(s)=0$. In the latter case, since we have excluded the possibility $s\in N^{-2/3}\mathbb{Z}$, it must hold that $\tilde{Z}^N$ is locally constant and equal to zero around s, so the component has size 1.

For large enough N, the intervals $\{[\bar{\alpha}^N(s_i),\bar{\beta}^N(s_i)]:i\in\Gamma_k\}$ are disjoint, and so

$$N^{-2/3}(C_1^{N,T},\ldots,C_k^{N,T})\succeq\left(\beta^N(s_1)-\alpha^N(s_1),\ldots,\beta^N\left(s_{|\Gamma_k|}\right)-\alpha^N\left(s_{|\Gamma_k|}\right),0,\ldots,0\right).$$

Since majorisation is preserved under limits (as the relation is a finite union of closed sets in $\mathbb{R}^k\times\mathbb{R}^k$), we obtain

$$(\bar{C}_1^T,\ldots,\bar{C}_k^T)\succeq\left(\bar{\beta}(s_1)-\bar{\alpha}(s_1),\ldots,\bar{\beta}\left(s_{|\Gamma_k|}\right)-\bar{\alpha}\left(s_{|\Gamma_k|}\right),0,\ldots,0\right).$$

So, combining with (2.70), we obtain

$$(\bar{C}_1^T,\ldots,\bar{C}_k^T)\succeq(C_1^T,\ldots,C_k^T), \tag{2.71}$$

which holds for every limit point $(\bar{C}_1^T,\ldots,\bar{C}_k^T)$ of $C^{N,T}$ on the event $\mathbf{\Psi}^T$ and so, in particular, $\mathbb{P}$-almost surely.

2.3.4 Stochastic sandwiching via excursions above δ

We now bound C^T stochastically (in the sense of weak majorisation) in the other direction.

Fix some $\delta > 0$. For any realisation of the path $\tilde{Z}^N$, the set $\mathcal{D}^{N,\delta,T} := \{s \in [0,T] : \tilde{Z}^N(s) > \delta\}$ is a finite union of left-closed, right-open intervals. Let $C^{N,\delta,T} = (C_1^{N,\delta,T}, \ldots, C_k^{N,\delta,T})$ be the sequence of the k largest lengths of those intervals which are contained within the support of some excursion of $\tilde{Z}^N$ (above zero) which has non-empty intersection with $[0,T]$. These are arranged in decreasing order, augmented with zeros if necessary. Certainly, for any δ, $C^{N,T} \succeq C^{N,\delta,T}$ for each trajectory of $\tilde{Z}^N$. We will show that C^T majorises limit points of $C^{N,\delta,T}$, again $\mathbb{P}$-almost surely.

Again, we work on the event Ψ^T. Then, consider $\mathcal{D}^T := \{s \in [0,T] : Z(s) > 0\}$, the collection of open intervals where the limit process Z is positive. On Ψ^T, for large enough N, we have $\tilde{Z}^N(s) \leq \delta/2$ whenever $Z(s) = 0$, and so $\mathcal{D}^{N,\delta,T} \subseteq \mathcal{D}^T$. Therefore the sequence of all interval lengths in $\mathcal{D}^{N,\delta,T}$ in non-increasing order is majorised by the corresponding ordered sequence of interval lengths in $\mathcal{D}^T$. So in particular

$$(C_1^T, \ldots, C_k^T) \succeq (C_1^{N,\delta,T}, \ldots, C_k^{N,\delta,T}),$$

for large enough N, and hence on Ψ^T any limit point $\bar{C}^{\delta,T}$ of $C^{N,\delta,T}$ satisfies

$$(C_1^T, \ldots, C_k^T) \succeq (\bar{C}_1^{\delta,T}, \ldots, \bar{C}_k^{\delta,T}).$$

By (2.63), the collection $(C^{N,T}, C^{N,\delta,T})_{N\geq 1}$ is tight in $\mathbb{R}^k \times \mathbb{R}^k$. Let $\bar{C}^T, \bar{C}^{\delta,T}$, be any joint weak limit of $C^{N,T}, C^{N,\delta,T}$. Since $\mathbb{P}\Big(\Psi^T\Big) = 1$, by combining with (2.71), we have shown that

$$\bar{C}^T \succeq_{st} C^T \succeq_{st} \bar{C}^{\delta,T}. \tag{2.72}$$

2.3.5 Comparing $C^{N,T}$ and $C^{N,\delta,T}$ via uniform trees

We will now show for small δ, any weak limits $\bar{C}^T, \bar{C}^{\delta,T}$ are close in distribution in the sense of the Lévy–Prohorov metric on $\mathbb{R}^k$. To do this, we have to bound above the probability that the exploration process drops below height $\delta N^{1/3}$ in the middle of an excursion above zero of width $\Theta(N^{2/3})$. The components of $\bar{G}(N,p)$ are, conditional on their sizes, uniform trees. We will apply results of Aldous in [3] to show that the large excursions of $\tilde{Z}^N$ are well-approximated by Brownian excursions. We then use this to bound the probability that $\tilde{Z}^N$ hits δ without hitting zero.

Let $\mathcal{T}_K$ be a uniform choice from the K^{K-2} unordered trees with vertex labels given by $[K]$. Then, let $1 = S_0^{\mathcal{T}_K}, S_1^{\mathcal{T}_K}, \dots, S_K^{\mathcal{T}_K} = 0$, be the corresponding breadth-first exploration process. The appropriate rescaling to consider is then $\tilde{S}^{\mathcal{T}_K}(s) := \frac{1}{\sqrt{K}} S^{\mathcal{T}_K}_{\lfloor Ks \rfloor}$, for $s \in [0,1]$. Recall from Proposition 1.13 that

$$\left(\tilde{S}^{\mathcal{T}_K}(s), s \in [0,1]\right) \xrightarrow{d} (B^{\mathrm{ex}}(s), s \in [0,1]), \tag{2.73}$$

where B^{ex} is a standard normalised Brownian excursion on $[0,1]$, and convergence is in the uniform topology.

We say the event $\chi^{N,T}(\delta, \epsilon, \gamma)$ holds if $\exists M, K \in \mathbb{Z}_{\geq 0}$ with $\frac{K}{N^{2/3}} \geq \gamma$, and $\frac{M}{N^{2/3}} \leq T$, such that $\{v_M, \dots, v_{M+K-1}\}$ is a component of $\bar{G}(N,p)$, and

$$\exists m \in [\epsilon K, (1-\epsilon)K] \text{ s.t. } \tilde{Z}^N\left(\tfrac{M+m}{N^{2/3}}\right) \leq \delta. \tag{2.74}$$

That is, $\bar{G}(N,p)$ has a component of size at least $\gamma N^{2/3}$ which is seen, at least partially, in the exploration process before time $TN^{2/3}$, and for which the exploration process takes a small value in the interior of the interval defining the component. Now, given any M, K, and conditional on the vertices $\{v_M, \dots, v_{M+K-1}\}$, and the statement that they form a component, the structure of this component is a uniform tree. That is,

$$(Z_M^N, \dots, Z_{M+K-1}^N) \stackrel{d}{=} (S_1^{\mathcal{T}_K}, \dots, S_K^{\mathcal{T}_K}).$$

Therefore the following processes on $s \in [0,1]$ can be identified in distribution:

$$\left(\tilde{Z}^N\left(\tfrac{M+sK}{N^{2/3}}\right)\right) = \left(N^{-1/3} Z^N_{\lfloor M+sK \rfloor}\right) \stackrel{d}{=} \left(N^{-1/3} S^{\mathcal{T}_K}_{\lfloor sK \rfloor}\right) = \left(\tfrac{K^{1/2}}{N^{1/3}} \tilde{S}^{\mathcal{T}_K}(s)\right).$$

Therefore, for every M, K, conditional on any choice of vertices $\{v_M, \ldots, v_{M+K-1}\}$, the probability that (2.74) holds is equal to the probability that

$$\inf_{s\in[\epsilon,1-\epsilon]} \tilde{S}^{\mathfrak{T}_K}(s) \leq \tfrac{N^{1/3}}{K^{1/2}}\delta. \tag{2.75}$$

By assumption $\frac{N^{1/3}}{K^{1/2}}\delta \leq \gamma^{-1/2}\delta$, and by (2.73), and the Portmanteau lemma,

$$\begin{aligned}\limsup_{K\to\infty} \mathbb{P}\left(\inf_{s\in[\epsilon,1-\epsilon]} \tilde{S}^{\mathfrak{T}_K}(s) \leq \gamma^{-1/2}\delta\right) &\leq \limsup_{K\to\infty} \mathbb{P}\left(\inf_{s\in[\epsilon,1-\epsilon]} \tilde{S}^{\mathfrak{T}_K}(s) < 2\gamma^{-1/2}\delta\right)\\ &\leq \mathbb{P}\left(\min_{s\in[\epsilon,1-\epsilon]} B^{\mathrm{ex}}(s) < 2\gamma^{-1/2}\delta\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{P}\Big(\chi^{N,T}(\delta,\epsilon,\gamma)\Big) &\leq \mathbb{E}\Big[\#\text{ cpts size } \geq \gamma N^{2/3} \text{ seen before } TN^{2/3} \text{ in } Z^N\Big]\\ &\quad\times \mathbb{P}\left(\min_{s\in[\epsilon,1-\epsilon]} B(s) < 2\gamma^{-1/2}\delta\right)\\ \limsup_{N\to\infty} \mathbb{P}\Big(\chi^{N,T}(\delta,\epsilon,\gamma)\Big) &\leq \left(\tfrac{T}{\gamma}+1\right)\mathbb{P}\left(\min_{s\in[\epsilon,1-\epsilon]} B(s) < 2\gamma^{-1/2}\delta\right). \end{aligned} \tag{2.76}$$

Given ϵ, γ, we can choose $\delta > 0$ so that the RHS of (2.76) is arbitrarily small. Now, fix some $\gamma > 2\epsilon$, and consider the event $\chi^{N,T}(\delta, \frac{\epsilon}{2\gamma}, \epsilon)$. Then, when $\chi^{N,T}(\delta, \frac{\epsilon}{2\gamma}, \epsilon)$, does not hold, for every component with size $K \geq \epsilon N^{2/3}$, there is a unique excursion of Z^N above $\delta N^{1/3}$ of length at least $K(1-\frac{\epsilon}{\gamma})$. We call such an excursion above $\delta N^{1/3}$ a *principal excursion*. If we also have $C_1^N \leq \gamma N^{2/3}$, then the length of any principal excursion is at least $K - \epsilon N^{2/3}$. Thus, any other excursion above $\delta N^{1/3}$ within the component of size K, has length at most $\epsilon N^{2/3}$.

So, consider any $i \leq k$ such that $C_1^{N,T} \geq \ldots C_i^{N,T} \geq \epsilon N^{2/3}$. Then, on $\chi^{N,T}(\delta, \frac{\epsilon}{2\gamma}, \epsilon)^c$ and $\{C_1^N \leq \gamma N^{2/3}\}$, at most $i-1$ elements of $C^{N,\delta,T}$ can be larger than $C_i^{N,T}$. These are the principal excursions obtained from each of $C_1^{N,T}, \ldots, C_{i-1}^{N,T}$. No other excursions above $\delta N^{2/3}$ obtained from $C_1^{N,T}, \ldots, C_{i-1}^{N,T}$ are relevant, since they have lengths at most $\epsilon N^{2/3}$. However, these principal excursions from $C_1^{N,T}, \ldots, C_i^{N,T}$ all have length at least $C_i^{N,T}(1-\frac{\epsilon}{\gamma})$. Thus we obtain

$$C_i^{N,T} \geq C_i^{N,\delta,T} \geq C_i^{N,T}(1-\tfrac{\epsilon}{\gamma}) \geq C_i^{N,T} - \epsilon N^{2/3}. \tag{2.77}$$

And so

$$\limsup_{N\to\infty} \mathbb{P}\left(\max_{i\in[k]}\left|C_i^{N,T} - C_i^{N,\delta,T}\right| > \epsilon\right) \leq \limsup_{N\to\infty} \mathbb{P}\Big(C_1^N > \gamma\Big) + \limsup_{N\to\infty} \mathbb{P}\Big(\chi^{N,T}(\delta, \tfrac{\epsilon}{2\gamma}, \epsilon)\Big).$$

For fixed $\epsilon > 0$, letting $\gamma \to \infty$ we can make the first term on the RHS small, and then by letting $\delta \downarrow 0$ we can make the second term small. In particular, we can demand

$$\limsup_{N\to\infty} \mathbb{P}\left(\max_{i\in[k]}\left|C_i^{N,T} - C_i^{N,\delta,T}\right| > \epsilon\right) \leq \epsilon. \tag{2.78}$$

Now, recall $\bar{C}^T, \bar{C}^{\delta,T}$ is some joint weak limit of $(C^{N,T}, C^{N,\delta,T})$. Let π be the usual Lévy–Prohorov metric for probability measures on $\mathbb{R}^k$, with respect to the ℓ_∞ norm on $\mathbb{R}^k$. From (2.72) and (2.78), we have for each $\epsilon > 0$,

$$\pi(\mathcal{L}(\bar{C}^T), \mathcal{L}(\bar{C}^{\delta,T})) \leq \epsilon, \quad \bar{C}^{\delta,T} \preceq C^T \preceq \bar{C}^T.$$

From this, it is easy to see that $\pi(\mathcal{L}(C^T), \mathcal{L}(\bar{C}^T)) \leq k\epsilon$. Since $\epsilon > 0$ is arbitrary, we find $\bar{C} \stackrel{d}{=} C$, and thus the required convergence in distibution (2.67) follows.

2.3.6 Proof of Theorem 2.7

In both the discrete exploration processes and the limiting SDEs, we would expect the k largest components/excursions to appear early. From (2.65),

$$\limsup_{T\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(\max_{i\in[k]}\left|C_i^{N,T} - C_i^N\right| \leq \epsilon\right) = 1.$$

By comparing the drifts, we can couple Z^λ and B^λ as defined in (2.3), such that $Z^\lambda(t) \leq B^\lambda(t)$ for all $t \geq 0$. The largest excursion of B^λ above zero is almost surely finite, and so the same holds for Z^λ. Thus

$$\limsup_{T\to\infty} \mathbb{P}\Big((C_1^T,\dots,C_k^T) = (C_1,\dots,C_k)\Big) = 1.$$

So we can lift (2.67) to conclude

$$(C_1^N,\dots,C_k^N) \quad \stackrel{d}{\to} \quad (C_1,\dots,C_k), \tag{2.79}$$

as $N\to\infty$.

Finally, we show convergence in $\ell^2_{\searrow}$. To lift (2.79), we require

$$\limsup_{N\to\infty} \mathbb{E}\left[\sum_{i=1}^\infty (C_i^{N,\lambda})^2\right] < \infty, \quad \mathbb{E}\left[\sum_{i=1}^\infty (C_i)^2\right] < \infty.$$

By Lemma 2.10, it suffices to show the first of these with for $G(N,p)$ instead of $\bar{G}(N,p)$. This appears as Corollary 5.2 in [34]. From the coupling of Z^λ and B^λ, it suffices to show the second bound for the excursion lengths of B^λ. This result is shown as Lemma 25 of Aldous's original result [5].

This completes the proof of Theorem 2.7.

2.4 Detailed combinatorial calculations

In this section we give proofs of three detailed lemmas required for the proof of Proposition 2.12 in the previous sections. We begin by restating and proving Lemma 2.17.

2.4.1 Proof of Lemma 2.17

Lemma. Fix $\lambda^- < \lambda^+ \in \mathbb{R}$. Given $p\in(0,1)$, let $\lambda = \lambda(N,p) = N^{1/3}(Np-1)$. Then

$$\mathbb{P}(G(N,p)\text{ acyclic}) = (1+o(1))g(\lambda)e^{3/4}\sqrt{2\pi}N^{-1/6}, \tag{2.80}$$

uniformly for $\lambda \in [\lambda^-, \lambda^+]$ as $N \to \infty$.

Proof. For this range of p, we will see that the sum in (2.19) is dominated by contributions on the scale $m = \frac{N}{2} + \frac{\lambda N^{2/3}}{2} + \Theta(N^{1/2})$. Shortly we will be required to approximate these relevant contributions in detail, but first we show that contributions from outside this regime vanish as $N \to \infty$. We consider those m for which

$$\left| m - \frac{N}{2} - \frac{\lambda N^{2/3}}{2} \right| \geq N^{3/5}.$$

Let $B \sim \mathrm{Bin}\Big(\binom{N}{2}, p\Big)$. Since $f(N, m) \leq \binom{\binom{N}{2}}{m}$,

$$(1-p)^{\binom{N}{2}} \left[\sum_{m=0}^{\lceil N/2 + \lambda N^{2/3}/2 - N^{3/5} \rceil} f(N,m) \Big(\tfrac{p}{1-p}\Big)^m + \sum_{\lfloor N/2 + \lambda N^{2/3}/2 + N^{3/5} \rfloor}^{N-1} f(N,m) \Big(\tfrac{p}{1-p}\Big)^m \right]$$

$$\begin{aligned} &\leq \mathbb{P}\Big(\big| B - \tbinom{N}{2} p \big| \geq N^{3/5} \Big) \\ &\leq \frac{\mathrm{Var}(B)}{N^{6/5}} \leq \frac{N^2 p}{2N^{6/5}} \leq \frac{1 + \lambda^+ N^{-1/3}}{2N^{1/5}} \ll N^{-1/6}. \end{aligned} \tag{2.81}$$

Here we used Chebyshev's inequality, which is sufficient for our purposes, but note that the probability of this moderate deviation event for B decays exponentially in some positive power of N.

Given $\lambda \in \mathbb{R}$ and $m \leq N \in \mathbb{N}$, define $x = x(N, m, \lambda) = \frac{\sqrt{2}}{N^{1/2}} \Big[m - \frac{N}{2} - \frac{\lambda N^{2/3}}{2} \Big]$. Then, we consider the set of m satisfying

$$\left| m - \frac{N}{2} - \frac{\lambda N^{2/3}}{2} \right| \leq N^{3/5}, \quad \text{that is, } |x| \leq \sqrt{2} N^{1/10}. \tag{2.82}$$

Thus

$$N - m = \tfrac{N}{2} - \tfrac{\lambda}{2} N^{2/3} - \tfrac{x}{\sqrt{2}} N^{1/2}, \quad \text{and so } \frac{2(N-m)}{N} = 1 - \lambda N^{-1/3} - \sqrt{2} x N^{-1/2}.$$

From this, we obtain

$$\log\left(\frac{2(N-m)}{N}\right) = -\lambda N^{-1/3} - \sqrt{2}xN^{-1/2} - \tfrac{\lambda^2}{2}N^{-2/3} - \sqrt{2}\lambda x N^{-5/6}$$
$$- \tfrac{\lambda^3}{3}N^{-1} - x^2N^{-1} + O(N^{-16/15}),$$

uniformly on the set of m defined at (2.82). In calculating the scale of this final error term, we use that $|x| \le \sqrt{2}N^{1/10}$. Then

$$\begin{aligned}(N-m)\log\left(\frac{2(N-m)}{N}\right) &= -\left[\tfrac{\lambda}{2}N^{2/3} + \tfrac{x}{\sqrt{2}}N^{1/2} + \tfrac{\lambda^2}{4}N^{1/3} + \tfrac{\lambda x}{\sqrt{2}}N^{1/6} + \tfrac{\lambda^3}{3} + \tfrac{x^2}{2}\right]\\
&\quad + \left[\tfrac{\lambda^2}{2}N^{1/3} + \tfrac{\lambda x}{\sqrt{2}}N^{1/6} + \tfrac{\lambda^3}{4}\right]\\
&\quad + \left[\tfrac{\lambda x}{\sqrt{2}}N^{1/6} + x^2\right] + O\left(N^{-1/15}\right)\\
&= -\tfrac{\lambda}{2}N^{2/3} - \tfrac{x}{\sqrt{2}}N^{1/2} + \tfrac{\lambda^2}{4}N^{1/3} + \tfrac{\lambda x}{\sqrt{2}}N^{1/6}\\
&\quad - \tfrac{\lambda^3}{12} + \tfrac{x^2}{2} + O\left(N^{-1/15}\right).\end{aligned}$$

We now return to (2.9) and use Stirling's approximation and the expression we have just shown, as well as continuity of g. Uniformly on the set of m in (2.82), (for which, recall, $N-m = (1+o(1))N/2$),

$$\begin{aligned}f(N,m) &= (1+o(1))\frac{\sqrt{2\pi}N^{N-1/6}}{2^{N-m}(N-m)!}g\left(\frac{2m-N}{N^{2/3}}\right),\\
&= (1+o(1))\frac{g(\lambda)\sqrt{2\pi}N^{N-1/6}}{2^{N-m}}\cdot\frac{1}{\sqrt{2\pi}\sqrt{N-m}}\left(\frac{e}{N-m}\right)^{N-m}\\
&= (1+o(1))g(\lambda)\sqrt{2}N^{m-2/3}\exp(N-m)\exp\left(-(N-m)\log\left(\tfrac{2(N-m)}{N}\right)\right)\\
&= (1+o(1))g(\lambda)\sqrt{2}N^{m-2/3}\exp\left(\tfrac{N}{2} - \tfrac{\lambda}{2}N^{2/3} - \tfrac{x}{\sqrt{2}}N^{1/2}\right)\\
&\qquad \times\exp\left(\tfrac{\lambda}{2}N^{2/3} + \tfrac{x}{\sqrt{2}}N^{1/2} - \tfrac{\lambda^2}{4}N^{1/3} - \tfrac{\lambda x}{\sqrt{2}}N^{1/6} + \tfrac{\lambda^3}{12} - \tfrac{x^2}{2}\right)\\
&= (1+o(1))g(\lambda)\sqrt{2}N^{m-2/3}\exp\left(\tfrac{N}{2} - \tfrac{\lambda^2}{4}N^{1/3} - \tfrac{\lambda x}{\sqrt{2}}N^{1/6} + \tfrac{\lambda^3}{12} - \tfrac{x^2}{2}\right). \qquad (2.83)\end{aligned}$$

Now, we have

$$\begin{aligned}\binom{N}{2}\log(1-p) &= \binom{N}{2}\left[-\tfrac{1+\lambda N^{-1/3}}{N} - \tfrac{1}{2}N^{-2} + O(N^{-7/3})\right]\\
&= -\tfrac{N}{2} - \tfrac{\lambda}{2}N^{2/3} + \tfrac{1}{4} + O\left(N^{-1/3}\right), \qquad (2.84)\end{aligned}$$

and also

$$\log\Big(\frac{Np}{1-p}\Big) = \log(1+\lambda N^{-1/3}) - \log(1-p)$$
$$= \lambda N^{-1/3} - \tfrac{\lambda^2}{2}N^{-2/3} + \tfrac{\lambda^3}{3}N^{-1} + N^{-1} + O\Big(N^{-4/3}\Big).$$

At this point, recall the definition

$$m = \tfrac{N}{2} + \tfrac{\lambda}{2}N^{2/3} + \tfrac{x}{\sqrt{2}}N^{1/2}.$$

So, uniformly on the set of m for which $|x| \leq \sqrt{2}N^{1/10}$, as before,

$$m\log\Big(\frac{Np}{1-p}\Big) = \Big[\tfrac{\lambda}{2}N^{2/3} - \tfrac{\lambda^2}{4}N^{1/3} + \tfrac{\lambda^3}{6} + \tfrac{1}{2}\Big] + \Big[\tfrac{\lambda^2}{2}N^{1/3} - \tfrac{\lambda^3}{4}\Big] + \tfrac{\lambda x}{\sqrt{2}}N^{1/6} + O\Big(N^{-1/6}\Big), \tag{2.85}$$

where each bracket corresponds to a term in the definition of m.

Therefore, combining (2.84) and (2.85), uniformly in the same sense,

$$(1-p)^{\binom{N}{2}}\Big(\tfrac{p}{1-p}\Big)^m = (1+o(1))N^{-m}\exp\Big(-\tfrac{N}{2} + \tfrac{\lambda^2}{4}N^{1/3} + \tfrac{\lambda x}{\sqrt{2}}N^{1/6} - \tfrac{\lambda^3}{12} + \tfrac{3}{4}\Big). \tag{2.86}$$

Combining (2.83) and (2.86), we obtain

$$(1-p)^{\binom{N}{2}}\Big(\tfrac{p}{1-p}\Big)^m f(N,m) = (1+o(1))g(\lambda)\sqrt{2}N^{-2/3}\exp\Big(-\tfrac{x^2}{2} + \tfrac{3}{4}\Big). \tag{2.87}$$

We now fix N and λ, and sum this quantity over the range of m given by (2.82). Recall that x is linear in m, with scaling factor $\frac{N^{1/2}}{\sqrt{2}}$, and so as $N \to \infty$, the sum of (2.87) over this range of m converges after rescaling to a integral. That is,

$$(1-p)^{\binom{N}{2}} \sum_{m=\lfloor N/2+\lambda N^{2/3}/2-N^{3/5}\rfloor}^{\lceil N/2+\lambda N^{2/3}/2+N^{3/5}\rceil} f(N,m)\Big(\tfrac{p}{1-p}\Big)^m$$
$$= (1+o(1))e^{3/4}g(\lambda)\sqrt{2}N^{-2/3} \sum_{m=\lfloor N/2+\lambda N^{2/3}/2-N^{3/5}\rfloor}^{\lceil N/2+\lambda N^{2/3}/2+N^{3/5}\rceil} e^{-x^2/2}$$

$$= (1+o(1))e^{3/4}g(\lambda)\sqrt{2}\frac{N^{-1/6}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x,$$
$$= (1+o(1))e^{3/4}g(\lambda)\sqrt{2\pi}N^{-1/6}.$$

Combining with (2.81), which showed that contributions to the sum (2.19) outside this range of m are $o(N^{-1/6})$, we obtain the required result. $\square$

2.4.2 Proof of Lemma 2.19

We restate Lemma 2.19.

Lemma. Fix constants $\lambda^-, \lambda^+, \epsilon, K, T$ as in Definition 2.18. Then,

$$\mathbb{P}(G(N',p) \in \mathcal{A}_{N',r,k}) = (1+o(1))g(\lambda - s - a)e^{3/4}N^{-5/6}ba^{-3/2} \qquad (2.25)$$
$$\times \exp\Big(-b(\lambda - s) - \tfrac{b^2}{2a} + \tfrac{(\lambda-s-a)^3-(\lambda-s)^3}{6}\Big),$$

uniformly on $(N', p, r, k) \in \Psi^N(\lambda^-, \lambda^+, \epsilon, K, T)$, as $N \to \infty$.

Proof. We will add the required uniformity in N' at the end of this proof. First, we show

$$\mathbb{P}(G(N,p) \in \mathcal{A}_{N,r,k}) = (1+o(1))g(\lambda - a)e^{3/4}N^{-5/6}ba^{-3/2} \qquad (2.88)$$
$$\times \exp\Big(-b\lambda - \tfrac{b^2}{2a} + \tfrac{(\lambda-a)^3-\lambda^3}{6}\Big),$$

uniformly on (p, r, k) such that $(N, p, r, k) \in \Psi^N(\lambda^-, \lambda^+, \epsilon, K, 0)$, as $N \to \infty$.

Subject to the constraint that vertices $1, \dots, r$ are in different tree components, with sum equal to k, there are $\binom{N-r}{k-r}$ ways to choose which remaining vertices are part of this stack forest. Given this choice, we can view the trees as rooted at the vertices $[r]$. In particular, Cayley's formula states that there are rk^{k-r-1} such labelled rooted forests. Hence

$$\mathbb{P}(G(N,p) \in \mathcal{A}_{N,r,k}) = (1-p)^{\binom{N}{2}}\binom{N-r}{k-r}\Big(\tfrac{p}{1-p}\Big)^{k-r} rk^{k-r-1}\sum_{m=0}^{N-k-1} f(N-k, m)\Big(\tfrac{p}{1-p}\Big)^m. \qquad (2.89)$$

By Lemma 2.17, uniformly on (p,k) and any (arbitrary) r such that $(N,p,r,k) \in \Psi^N(\lambda^-,\lambda^+,\epsilon,K,0)$,

$$(1-p)^{\binom{N-k}{2}} \sum_{m=0}^{N-k-1} f(N-k,m)\Big(\tfrac{p}{1-p}\Big)^m = (1+o(1))g(\lambda(N-k,p))e^{3/4}\sqrt{2\pi}N^{-1/6}.$$

Recall that this final sum is, up to a power of $(1-p)$, the probability that $G(N-k,p)$ is acyclic. We also have

$$\begin{aligned}\lambda(N-k,p) &= (N-aN^{2/3})^{1/3}\Big[(N-aN^{2/3})p-1\Big]\\ &= (1+o(1))N^{1/3}\Big[(Np-1)-aN^{-1/3}\Big]\\ &= (1+o(1))[\lambda(N,p)-a+o(1)].\end{aligned}$$

So, again uniformly on (p,r,k) such that $(N,p,r,k) \in \Psi^N(\lambda^-,\lambda^+,\epsilon,K,0)$,

$$(1-p)^{\binom{N-k}{2}} \sum_{m=0}^{N-k-1} f(N-k,m)\Big(\tfrac{p}{1-p}\Big)^m = (1+o(1))g(\lambda-a)e^{3/4}\sqrt{2\pi}N^{-1/6}. \quad (2.90)$$

We now carefully address the other terms in (2.89), starting with $(1-p)^{\binom{N}{2}-\binom{N-k}{2}}\left(\frac{p}{1-p}\right)^{k-r}$. Recall that $Np = 1+\lambda N^{-1/3}$. Firstly

$$\begin{aligned}\log\Big[\Big(1+\lambda N^{-1/3}\Big)^{k-r}\Big] &= \Big[aN^{2/3}-bN^{1/3}\Big]\Big[\lambda N^{-1/3}-\tfrac{\lambda^2}{2}N^{-2/3}+O\Big(N^{-1}\Big)\Big]\\ &= \lambda a N^{1/3}-\lambda b-\tfrac{\lambda^2 a}{2}+O\Big(N^{-1/3}\Big).\end{aligned}$$

Also

$$\begin{aligned}\tbinom{N}{2}-\tbinom{N-k}{2}-k+r &= \tfrac{N^2}{2}-\tfrac{(N-k)^2}{2}+\tfrac{k}{2}-k+r\\ &= Nk-\tfrac{k^2}{2}+O\Big(N^{2/3}\Big)\\ &= aN^{5/3}-\tfrac{a^2}{2}N^{4/3}+O\Big(N^{2/3}\Big),\end{aligned}$$

from which

$$\log\Big[(1-p)^{\binom{N}{2}-\binom{N-k}{2}-k+r}\Big] = \log\Big[\Big(1-N^{-1}-\lambda N^{-4/3}\Big)^{\binom{N}{2}-\binom{N-k}{2}-k+r}\Big]$$

$$= \Big[aN^{5/3} - \tfrac{a^2}{2}N^{4/3} + O\Big(N^{2/3}\Big)\Big]\Big[-N^{-1} - \lambda N^{-4/3} + O\Big(N^{-2}\Big)\Big]$$
$$= -aN^{2/3} - \lambda a N^{1/3} + \tfrac{a^2}{2}N^{1/3} + \tfrac{\lambda a^2}{2} + O\Big(N^{-1/3}\Big).$$

From this,

$$\begin{aligned}&(1-p)^{\binom{N}{2}-\binom{N-k}{2}}\Big(\tfrac{p}{1-p}\Big)^{k-r}\\ &\quad = (1+o(1))N^{-(k-r)}\exp\Big(-aN^{2/3} + \tfrac{a^2}{2}N^{1/3} - \lambda b + \tfrac{\lambda a}{2}(a-\lambda)\Big).\end{aligned} \tag{2.91}$$

Turning now to the binomial coefficent $\binom{N-r}{k-r}$ in (2.89), we treat each factorial separately. First observe that

$$\begin{aligned}\log\Big[\Big(1-bN^{-2/3}\Big)^{N-bN^{1/3}}\Big] &= \Big[N - bN^{1/3}\Big]\Big[-bN^{-2/3} + O\Big(N^{-4/3}\Big)\Big]\\ &= -bN^{1/3} + O(N^{-1/3}).\\ \log\Big[\Big(1-aN^{-1/3}\Big)^{N-aN^{2/3}}\Big] &= \Big[N - aN^{2/3}\Big]\Big[-aN^{-1/3} - \tfrac{a^2}{2}N^{-2/3} - \tfrac{a^3}{3}N^{-1} + O\Big(N^{-4/3}\Big)\Big]\\ &= -aN^{2/3} + \tfrac{a^2}{2}N^{1/3} + \tfrac{a^3}{6} + O\Big(N^{-1/3}\Big)\\ \log\Big[\Big(1-\tfrac{b}{a}N^{-1/3}\Big)^{aN^{2/3}-bN^{1/3}}\Big] &= \Big[aN^{2/3} - bN^{1/3}\Big]\Big[-\tfrac{b}{a}N^{-1/3} - \tfrac{b^2}{2a^2}N^{-2/3} + O\Big(N^{-1}\Big)\Big]\\ &= -bN^{1/3} + \tfrac{b^2}{2a} + O\Big(N^{-1/3}\Big).\end{aligned}$$

Then Stirling's approximation gives

$$\begin{aligned}\Big(N - bN^{1/3}\Big)! &= (1+o(1))\frac{\sqrt{2\pi N}}{e^{N-bN^{1/3}}}\Big(N - bN^{1/3}\Big)^{N-bN^{1/3}}\\ &= (1+o(1))\frac{\sqrt{2\pi N}}{e^{N-bN^{1/3}}}N^{N-bN^{1/3}}\exp\Big(-bN^{1/3}\Big)\\ \Big(N - aN^{2/3}\Big)! &= (1+o(1))\frac{\sqrt{2\pi N}}{e^{N-aN^{2/3}}}\Big(N - aN^{2/3}\Big)^{N-aN^{2/3}}\\ &= (1+o(1))\frac{\sqrt{2\pi N}}{e^{N-aN^{2/3}}}N^{N-aN^{2/3}}\exp\Big(-aN^{2/3} + \tfrac{a^2}{2}N^{1/3} + \tfrac{a^3}{6}\Big)\\ \Big(aN^{2/3} - bN^{1/3}\Big)! &= (1+o(1))\frac{\sqrt{2\pi}\sqrt{aN^{2/3}}}{e^{aN^{2/3}-bN^{1/3}}}\Big(aN^{2/3} - bN^{1/3}\Big)^{aN^{2/3}-bN^{1/3}}\\ &= (1+o(1))\frac{\sqrt{2\pi}\sqrt{aN^{2/3}}}{e^{aN^{2/3}-bN^{1/3}}}a^{aN^{2/3}-bN^{1/3}}N^{\frac{2}{3}\left[aN^{2/3}-bN^{1/3}\right]}\\ &\qquad \exp\Big(-bN^{1/3} + \tfrac{b^2}{2a}\Big).\end{aligned}$$

So we obtain

$$\binom{N-r}{k-r} = (1+o(1))\tfrac{1}{\sqrt{2\pi}}a^{-(aN^{2/3}-bN^{1/3}+1/2)}N^{\frac{1}{3}(aN^{2/3}-bN^{1/3}-1)} \\ \times \exp\Big(aN^{2/3} - \tfrac{a^2}{2}N^{1/3} - \tfrac{b^2}{2a} - \tfrac{a^3}{6}\Big). \tag{2.92}$$

The final ingredient of (2.89) is the term

$$rk^{k-r-1} = ba^{aN^{2/3}-bN^{1/3}-1}N^{\frac{2}{3}\left[aN^{2/3}-bN^{1/3}\right]-\frac{1}{3}}. \tag{2.93}$$

To recover (2.89), we study the product of (2.90), (2.91), (2.92) and (2.93). Note that $\exp\Big(-\frac{\lambda^2 a}{2} + \frac{\lambda a^2}{2} - \frac{a^3}{6}\Big) = \exp\Big(\frac{(\lambda-a)^3-\lambda^3}{6}\Big)$. So we can treat all of the terms in (2.89) uniformly on (p,r,k) such that $(N,p,r,k) \in \Psi^N(\lambda^-,\lambda^+,\epsilon,K,0)$, as $N \to \infty$ and obtain (2.88) as required.

We now finish the proof of (2.25), where in addition we require a uniform estimate over $N' \in [N - TN^{2/3}, N]$. We consider $(N',p,r,k) \in \Psi^N(\lambda^-,\lambda^+,\epsilon,K,T)$ as $N \to \infty$. Observe that

$$\lambda' := \lambda(N',p) = (1+o(1))(\lambda(N,p)-s), \qquad N' = (1+o(1))N, \tag{2.94}$$

$$b' := b(N',r) = (1+o(1))b(N,r), \quad a' = a(N',k) = (1+o(1))a(N,k). \tag{2.95}$$

Now fix $\delta \in (0,\epsilon)$. Then, for large enough N,

$$\begin{aligned} & (N',p,r,k) \in \Psi^N(\lambda^-,\lambda^+,\epsilon,K,T) \\ \Rightarrow \quad & (N',p,r,k) \in \Psi^{N'}(\lambda^- - T - \delta, \lambda^+ + \delta, \epsilon - \delta, K + \delta, 0). \end{aligned} \tag{2.96}$$

Certainly $N - TN^{2/3} \to \infty$ as $N \to \infty$, so by (2.88) and (2.96),

$$\mathbb{P}(G(N',p) \in \mathcal{A}_{N,r,k}) = (1+o(1))g(\lambda'-a')e^{3/4}N'^{-5/6}b'a'^{-3/2} \\ \times \exp\Big(-b'\lambda' - \tfrac{b'^2}{2a'} + \tfrac{(\lambda'-a')^3-\lambda'^3}{6}\Big),$$

uniformly on $(N', p, r, k) \in \Psi^N(\lambda^-, \lambda^+, \epsilon, K, T)$ as $N \to \infty$. Finally, using (2.94), (2.95), and the fact that g is uniformly continuous, we may conclude

$$\mathbb{P}\big(G(N', p) \in \mathcal{A}_{N',r,k}\big) = (1 + o(1))g(\lambda - s - a)e^{3/4}N^{-5/6}ba^{-3/2}$$

$$\times \exp\Big(-b(\lambda - s) - \tfrac{b^2}{2a} + \tfrac{(\lambda-a-s)^3-(\lambda-s)^3}{6}\Big),$$

as required, uniformly on $(N', p, r, k) \in \Psi^N(\lambda^-, \lambda^+, \epsilon, K, T)$. □

2.4.3 Proof of Lemma 2.21

We now restate and prove Lemma 2.21, which was used in the proof of Lemma 2.20.

Lemma. Given the same constants as in Lemma 2.20, there exist constants $M < \infty$ and $\gamma > 0$ such that

$$\frac{(k+1)\mathbb{P}\big(G(N', p) \in \mathcal{A}_{N',r,k+1}\big)}{k\mathbb{P}\big(G(N', p) \in \mathcal{A}_{N',r,k}\big)} \leq 1 - \gamma N^{-2/3}, \qquad (2.31)$$

for large enough N, whenever $(N', p, r) \in \bar{\Psi}_0^N(\lambda^-, \lambda^+, K, T)$ and $k \in [MN^{2/3}, N' - 1]$.

Proof. Again, we will use (2.89), which for convenience we recall here.

$$\begin{aligned}\mathbb{P}(G(N,p) \in \mathcal{A}_{N,r,k}) &= (1-p)^{\binom{N}{2}}\tbinom{N-r}{k-r}\Big(\tfrac{p}{1-p}\Big)^{k-r} rk^{k-r-1}\sum_{m=0}^{N-k-1} f(N-k,m)\Big(\tfrac{p}{1-p}\Big)^m \\ &= (1-p)^{\binom{N}{2}-\binom{N-k}{2}}\tbinom{N-r}{k-r}\Big(\tfrac{p}{1-p}\Big)^{k-r} rk^{k-r-1}F(N-k,p).\end{aligned}$$

We apply this to (2.31) (with N replaced by N'). Note that $\binom{N'-r}{k+1-r}/\binom{N'-r}{k-r} = \frac{N'-k}{k+1-r}$, and $\binom{N'-k}{2} - \binom{N'-k-1}{2} = N' - k - 1$. We obtain

$$\begin{aligned}&\frac{(k+1)\mathbb{P}\big(G(N', p) \in \mathcal{A}_{N',r,k+1}\big)}{k\mathbb{P}\big(G(N', p) \in \mathcal{A}_{N',r,k}\big)} \\ &\quad= \frac{(k+1)(1-p)^{-\binom{N'-k-1}{2}}\binom{N'-r}{k+1-r}\Big(\frac{p}{1-p}\Big)r(k+1)^{k-r}F(N'-k-1,p)}{k(1-p)^{-\binom{N'-k}{2}}\binom{N'-r}{k-r}rk^{k-r-1}F(N'-k,p)}\end{aligned}$$

$$= \frac{k+1}{k+1-r} \cdot (1-p)^{N'-k-2} \cdot \frac{N'-k}{N} \cdot \Big(1+\lambda N^{-1/3}\Big) \cdot \Big(\frac{k+1}{k}\Big)^{k-r} \cdot \frac{F(N'-k-1,p)}{F(N'-k,p)}. \tag{2.97}$$

We proceed in two parts. First we control the ratio of the $F(N'-k,p)$ terms using (2.20). Then, we control the ratio of the remaining terms with an elementary but long Taylor expansion.

First, note that from the second inequality in (2.20), that for $k \le N'-1$,

$$1 - \frac{F(N'-k,p)}{F(N'-k-1,p)} \le \frac{1}{2}(N'-k-1)p^2 \mathbb{E}\Big[|C^{N'-k-1,p}(v)|\Big].$$

where $|C^{N,p}(v)|$ is the size of the component containing a uniformly chosen vertex v in $G(N,p)$. Now, via (2.94),

$$\limsup_{N\to\infty} \lambda\Big(N - \lfloor MN^{2/3}\rfloor, p\Big) \le \lambda^+ - M.$$

When $k \ge MN^{2/3}$, we have

$$N^{-1/3}\mathbb{E}\Big[|C^{N'-k-1,p}(v)|\Big] \le N^{-1/3}\mathbb{E}\Big[|C^{N-\lfloor MN^{2/3}\rfloor,p}(v)|\Big],$$

and so from (2.10),

$$\limsup_{N\to\infty} \sup_{\substack{N'\in[N-TN^{2/3},N]\\ k\ge MN^{2/3}}} N^{-1/3}\mathbb{E}\Big[|C^{N'-k-1,p}(v)|\Big] \le \Theta^{\lambda^+-M}.$$

We obtain

$$\limsup_{N\to\infty} \sup_{\substack{N'\in[N-TN^{2/3},N]\\ \lambda(N,p)\in[\lambda^-,\lambda^+]\\ 0\le k\le N'-1}} N^{2/3}\Big[1 - \frac{F(N'-k,p)}{F(N'-k-1,p)}\Big] \le \frac{1}{2}\Theta^{\lambda^+-M},$$

from which it follows that

$$\limsup_{N\to\infty} \sup_{\substack{N'\in[N-TN^{2/3},N]\\ \lambda(N,p)\in[\lambda^-,\lambda^+]\\ 0\le k\le N'-1}} N^{2/3}\Big[\frac{F(N'-k-1,p)}{F(N'-k,p)} - 1\Big] \le \frac{1}{2}\Theta^{\lambda^+-M}. \tag{2.98}$$

We now treat the remaining terms in the ratio (2.97), that is

$$\frac{k+1}{k+1-r}\cdot(1-p)^{N'-k-2}\cdot\frac{N'-k}{N}\cdot\Big(1+\lambda N^{-1/3}\Big)\cdot\Big(\frac{k+1}{k}\Big)^{k-r}.$$

We split the calculation into several steps. Recall the rescalings $a=\frac{k}{N^{2/3}}$ and $b=\frac{r}{N^{1/3}}$. Since we assume $k\geq MN^{2/3}$, we have $\frac{1}{a}=O(1)$.

$$\begin{aligned}
\log\Big(\frac{k+1}{k+1-r}\Big)&=-\log\Big(1-\tfrac{r}{k+1}\Big)=\tfrac{r}{k+1}+\tfrac12\Big(\tfrac{r}{k+1}\Big)^2+O(N^{-1})\\
&=\tfrac{b}{a}N^{-1/3}+\tfrac{b^2}{2a^2}N^{-2/3}+O\Big(N^{-1}\Big),\\
\log\Big(1+\lambda N^{-1/3}\Big)&=\lambda N^{-1/3}-\tfrac{\lambda^2}{2}N^{-2/3}+O(N^{-1}),\\
\log\Bigg[\Big(\frac{k+1}{k}\Big)^{k-r}\Bigg]&=\Big[aN^{2/3}-bN^{1/3}\Big]\Big[\tfrac{1}{a}N^{-2/3}-\tfrac{1}{2a^2}N^{-4/3}+O\Big(N^{-2}\Big)\Big]\\
&=1-\tfrac{b}{a}N^{-1/3}-\tfrac{1}{2a}N^{-2/3}+O\Big(N^{-1}\Big).
\end{aligned}$$

The final two terms in the product require extra care, because there is no finite upper bound on a. However, since $a\leq N^{1/3}$, we can still handle the error in the following term:

$$\begin{aligned}
\log\Big[(1-p)^{N'-k-2}\Big]&=\Big[N-(s+a)N^{2/3}-2\Big]\Big[-N^{-1}-\lambda N^{-4/3}+O\Big(N^{-2}\Big)\Big]\\
&=-1+(s-\lambda+a)N^{-1/3}+\lambda(a+s)N^{-2/3}+O(N^{-1}).
\end{aligned}$$

Finally, we have

$$\log\Big(\frac{N'-k}{N}\Big)=\log\Big(1-sN^{-1/3}-aN^{-1/3}\Big)\leq-(a+s)N^{-1/3}-\tfrac12(a+s)^2N^{-2/3}.$$

So there exists a constant $C=C(\lambda^-,\lambda^+,\epsilon,K,T)<\infty$ such that

$$\begin{aligned}
&\log\Bigg[\frac{k+1}{k+1-r}\cdot(1-p)^{N'-k-2}\cdot\frac{N'-k}{N}\cdot\Big(1+\lambda N^{-1/3}\Big)\cdot\Big(\frac{k+1}{k}\Big)^{k-r}\Bigg]\\
&\qquad\leq N^{-2/3}\Bigg[-\frac12(\lambda-(a+s))^2+\frac{b^2}{2a^2}-\frac{1}{2a}\Bigg]+\frac{C}{N},
\end{aligned}\tag{2.99}$$

uniformly on $(N',p,r)\in\Psi_0^N(\lambda^-,\lambda^+,\epsilon,K,T)$ and $k\geq MN^{2/3}$, as $N\to\infty$. Recall that $b\in[\epsilon,K]$, and that $k\geq MN^{2/3}$ is equivalent to $a\geq M$. So for large enough M, the term $\frac{b^2}{2a^2}$ is dominated by the term $-\frac{1}{2a}$ in (2.99). So it holds that for large enough N,

$$\frac{k+1}{k+1-r}\cdot(1-p)^{N'-k-2}\cdot\frac{N'-k}{N}\cdot\left(1+\lambda N^{-1/3}\right)\cdot\left(\frac{k+1}{k}\right)^{k-r}\leq 1-\frac{1}{3K}N^{-2/3}.$$

Using Proposition 2.11, we now also demand that M be large enough that $\Theta^{\lambda^+-M}\leq\frac{1}{6K}$. So combining with (2.98), we can now approximate the LHS of (2.31) as required. Now take $\gamma\in(0,\frac{1}{6K})$, and we find that for large enough N

$$\frac{(k+1)\mathbb{P}\big(G(N,p)\in\mathcal{A}_{N',r,k+1}\big)}{k\mathbb{P}\big(G(N,p)\in\mathcal{A}_{N',r,k}\big)}\leq 1-\gamma N^{-2/3}.$$

□

Chapter 3

Large components in inhomogeneous random graphs

In Chapter 5 we will consider a version of mean-field frozen percolation where the vertices have types drawn from a finite set. In this chapter, before introducing the frozen percolation dynamics, we prove some results about a certain class of random graphs where the vertices have types.

This chapter has four sections. First, we define the inhomogeneous random graph model (IRG), and review Bollobás, Janson and Riordan's results about its phase transition, its giant component, and the natural connection to a multitype branching process. In Section 3.2, we present a new exponential tail bound for the size of a component in an IRG close to criticality. We use this in Section 3.3, where we prove a concentration result for the proportion of types in any large component of such an IRG close to criticality. During this and subsequent chapters, we will require some technical results about non-negative matrices and their eigenvectors. To avoid breaking the flow of the otherwise mostly probabilistic arguments, we collect the proofs of these technical results in Section 3.4.

3.1 Background, definitions and notation

3.1.1 Motivation

In the standard Erdős–Rényi random graph $G(N, c/N)$, the degree of a given vertex has a binomial distribution. Furthermore, for large N, the degrees of different vertices are almost independent, and so the empirical degree distribution is approximately Poisson(c). Degree sequences in networks observed in the real-world are much less homogeneous. In many contexts, such as modelling the worldwide web, we might imagine a small collection of 'hubs' with large degree, connected to each other, and supporting sparsely-connected outliers.

To model these effects, it is natural to consider relaxing the independence between different edges, or the assumption that vertices have the same underlying local degree distribution. Albert and Barabási [2] studied heuristics for several such models, and many subsequent papers have treated their mathematical properties.

3.1.2 Inhomogeneous random graphs with k types

This model was introduced by Söderberg [64] and studied in a version with more general type-spaces by Bollobás, Janson and Riordan [15].

Throughout, we fix a positive integer k. A *graph with k types* is a graph $G = (V, E)$ together with a *type function*, $\mathrm{type} : V \to [k]$.

Definition 3.1. A $k \times k$ symmetric matrix κ with non-negative real entries is a *kernel*. If κ has positive entries, then it is a *positive kernel*. We refer to the sets of kernels and positive kernels as $\mathbb{R}_{\geq 0}^{k\times k}$ and $\mathbb{R}_{+}^{k\times k}$ respectively. Similarly, the set of kernels with entries in $[a, b]$ is $[a, b]^{k\times k}$. We use the abbreviations $\kappa_{\max} := \max_{i,j\in[k]} \kappa_{i,j}$ and $\kappa_{\min} := \min_{i,j\in[k]} \kappa_{i,j}$.

When we consider the proportions of vertices of each type, we will refer to the sets

$$\Pi_1 := \left\{ \pi \in \mathbb{R}_{\geq 0}^{k} \,:\, \sum_{i\in[k]} \pi_i = 1 \right\}, \quad \Pi_{\leq 1} := \left\{ \pi \in \mathbb{R}_{\geq 0}^{k} \,:\, \sum_{i\in[k]} \pi_i \leq 1 \right\}, \tag{3.1}$$

of probability distributions and sub-distributions on $[k]$.

Note. We will regularly write $\kappa \le \kappa'$ for kernels and similarly $v \le v'$ for vectors. Unless specified otherwise, this ordering is always taken to be coordinate-wise.

Definition 3.2. For each $N \in \mathbb{N}$, $p = (p_1, \dots, p_k) \in \mathbb{N}_0^k$ and κ a kernel, the *inhomogeneous random graph* $G^N(p, \kappa)$ is a random graph with k types defined as follows:

- $G^N(p, \kappa)$ has vertex set $\left\{1, 2, \dots, \sum_{i=1}^k p_i\right\}$.
- The type function is chosen uniformly at random from the $\binom{\sum p_i}{p_1, \dots, p_k}$ functions $f : [\sum p_i] \to [k]$ such that $|f^{-1}(\{i\})| = p_i$ for each i.
- Conditional on the type function, each edge vw (with $v \neq w \in [\sum p_i]$) is present with probability

$$1 - \exp(-\kappa_{\mathrm{type}(v),\mathrm{type}(w)}/N),$$

  independently of all other pairs.

Remark. Note that the edge probability $1 - \exp(-\kappa_{i,j}/N) \approx \kappa_{i,j}/N$. While most results hold with this simpler alternative definition, the one taken here avoids the necessity to ensure $N \ge \kappa_{\max}$, and, importantly, fits the natural extension to a graph process, where edges appear at independent exponentially distributed times.

Remark. We are using slightly different notation to [15]. On many occasions, we will assume $\sum p_i = N$, whence p/N is a probability distribution. In the next chapter, the main process we study will involve varying p/N, a vector which records the proportions of each type. This corresponds to the measure on the (generalised) ground space as defined in [15], where it is mostly taken to be fixed.

3.1.3 Branching process analogy and criticality of IRGs

As introduced in Section 1.1.1, the classical Erdős–Rényi random graph $G(N, \frac{c}{N})$ contains a giant component with high probability exactly when $c > 1$. This corresponds directly to the regime in which a Galton–Watson branching process with offspring distribution Poisson(c) has positive survival probability.

We now discuss an analogous correspondence between the inhomogeneous random graph and a related multitype Poisson branching process.

Definition 3.3. Given a kernel $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$ and a measure $\pi \in \mathbb{R}_{\geq 0}^{k}$, we define the $k \times k$ matrix $\kappa \circ \pi$ by

$$[\kappa \circ \pi]_{i,j} := \kappa_{i,j}\pi_j.$$

The number of type j neighbours of a type i parent in $G^N(p, \kappa)$ has the distribution $\mathrm{Bin}\Big(p_j, 1 - e^{-\kappa_{i,j}/N}\Big)$ when $i \neq j$, and $\mathrm{Bin}\Big(p_j - 1, 1 - e^{-\kappa_{i,i}/N}\Big)$ when $i = j$. Therefore, when N is large, the expectation of this quantity is approximately $[\kappa \circ p/N]_{i,j}$.

We now define a multitype analogue of Galton–Watson trees with Poisson offspring distributions.

Definition 3.4. For $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$ and $\pi \in \Pi_{\leq 1}$, we define a *branching process tree with k types* to be either empty or a random ordered, rooted tree $\Xi^{\pi,\kappa} \subseteq \mathcal{U}$ together with a map called type : $\Xi^{\pi,\kappa} \to [k]$ as follows:

- With probability $1 - \sum_{i=1}^{k} \pi_i$, set $\Xi^{\pi,\kappa} = \varnothing$.
- For each i, with probability π_i, declare $\varnothing \in \Xi^{\pi,\kappa}, \mathrm{type}(\varnothing) = i$.
- For each $\ell \geq 0$, we construct the tree at generation $\ell + 1$ recursively and independently from the tree at generation ℓ. For every $u = (u_1, \ldots, u_\ell) \in \mathcal{U}$ that is already in $\Xi^{\kappa,\pi}$, and has $\mathrm{type}(u) = i$, sample independent random variables $c_1(u), \ldots, c_k(u)$, such that $c_j(u) \sim \mathrm{Po}(\kappa_{i,j}\pi_j)$. Define the number of children $c(u)$ of u in $\Xi^{\pi,\kappa}$ to be $\sum_{j\in[k]} c_j(u)$. Then, for each $j \in [k]$, set

$$\mathrm{type}(u_1, \ldots, u_\ell, w) = j, \quad \text{for } w \in \left[1 + \sum_{j'=1}^{j-1} c_{j'}(u), \sum_{j'=1}^{j} c_{j'}(u)\right].$$

  That is, u has $c_j(u)$ children with type j.

Note. It is worth emphasising that the cases $\Xi^{\kappa,\pi} = \varnothing$ and $\Xi^{\kappa,\pi} = \{\varnothing\}$ are different. The former is empty, while the latter has a root and no other vertices.

Let $|\Xi^{\pi,\kappa}|$ be the total population size of $\Xi^{\pi,\kappa}$, and describe the event $\{|\Xi^{\pi,\kappa}| = \infty\}$ as *survival*, and the event $\{|\Xi^{\pi,\kappa}| < \infty\}$ as *extinction*.

Although the authors use slightly different terminology, Bollobás, Janson and Riordan [15] show that for $\pi \in \Pi_1$, and $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$, the branching process $\Xi^{\pi,\kappa}$ is the local limit of $G^N(p, \kappa)$, whenever $p/N \to \pi$ as $N \to \infty$.

Definition 3.5. It follows by the Perron–Frobenius theorem that for any positive matrix $A \in \mathbb{R}_{+}^{k\times k}$, there exists $\rho(A) \in \mathbb{R}_+$ such that $\rho(A)$ is a simple eigenvalue of A, and all other eigenvalues $\rho' \in \mathbb{C}$ satisfy $|\rho'| < \rho(A)$. This principal eigenvalue $\rho(A)$ is called the *Perron root* [57]. Furthermore, the left-eigenvector (or right-eigenvector) corresponding to $\rho(A)$ can be normalised such that all its components are positive.

We define $\mu(A), \nu(A)$ to be these principal left- and right-eigenvectors respectively, normalised such that $\sum_{i=1}^{k} \mu_i(A) = \sum_{i=1}^{k} \nu_i(A) = 1$.

Remark. Some of the Perron–Frobenius theory can be extended to non-negative matrices [29]. In particular, for $A \in \mathbb{R}_{\geq 0}^{k\times k}$ it remains true that there exists an eigenvalue $\rho(A) \in \mathbb{R}_{\geq 0}$ for which a corresponding eigenvector is non-negative, and in this case all other eigenvalues $\rho' \in \mathbb{C}$ satisfy $|\rho'| \leq \rho(A)$.

Definition 3.6. Following [15], we say that both the branching process $\Xi^{\pi,\kappa}$ and the random graph $G^N(p, \kappa)$ are *subcritical* if $\rho(\kappa \circ \pi) < 1$, *critical* if $\rho(\kappa \circ \pi) = 1$, and *supercritical* if $\rho(\kappa \circ \pi) > 1$, where π is defined to be p/N, as before.

For the branching process, the eigenvalue $\rho(\kappa \circ \pi)$ plays the same role as the mean of the offspring distribution for controlling the survival probability in the original Galton–Watson process, as described in Proposition 1.6. This is formalised for the multitype setting shortly in Proposition 3.7, via Mode [52]. Similarly, $\rho(\kappa \circ \pi)$ is the analogue of c in the original Erdős–Rényi graph $G(N, c/N)$, for controlling the existence of a giant component. This is formalised in Theorem 3.8 shortly.

Proposition 3.7. [52, §1 Theorem 7.1] Let $\pi \in \Pi_{\leq 1}$ be positive, and let κ be a positive kernel. Then,

- if $\rho(\kappa \circ \pi) \leq 1$, then $\mathbb{P}(|\Xi^{\pi,\kappa}| = \infty) = 0$;

- if $\rho(\kappa \circ \pi) > 1$, then $\zeta_i^{\pi,\kappa} := \mathbb{P}(|\Xi^{\pi,\kappa}| = \infty \,|\, \text{type(root)} = i) > 0$, for all $i \in [k]$.

That is, there is a positive probability of survival iff $\rho(\kappa \circ \pi) > 1$. Furthermore, analogously to (1.2), $\zeta^{\pi,\kappa}$ is the maximal solution to

$$\zeta_j^{\pi,\kappa} = 1 - \exp(-[(\kappa \circ \pi)\zeta^{\pi,\kappa}]_j). \tag{3.2}$$

Remark. Subsequent chapters of [52] treat various cases where $\kappa \circ \pi$ has some zero entries but the result is the same. Here and in Chapter 5, we will typically assume kernels are positive, so that the uniqueness of the principal left-eigenvector is automatic.

The heuristic for the existence of a giant component is similar to the original monotype case, which we discussed in the final example of Section 1.2.3. The probability that a vertex is contained in a giant component approaches the probability that the branching process survives. Furthermore, it can be shown that the graph is exponentially unlikely to include multiple giant components, or a positive proportion of vertices contained in 'large but not giant' components.

In this chapter, for a graph G, we let $L_1(G)$ be the size of the largest component in G, and $\mathcal{L}_1(G)$ be a component with size $L_1(G)$. At no stage in this chapter will it matter how the tie-breaking is applied if necessary. We may now state Bollobás, Janson and Riordan's main theorem about the giant component in a large IRG.

THEOREM 3.8. [15, Theorems 3.1, 9.10] Fix a positive kernel $\kappa \in \mathbb{R}_+^{k\times k}$ and probability distribution $\pi \in \Pi_1$. Let $p^N \in \mathbb{N}_0^k$ satisfy $\sum_{i\in[k]} p_i^N = N$ and $p^N/N \to \pi$, as $N \to \infty$. Then, with high probability as $N \to \infty$,

$$L_1\Big(G^N(p^N,\kappa)\Big) = \begin{cases} o(N) & \rho(\kappa\circ\pi) \le 1 \\ \Theta(N) & \rho(\kappa\circ\pi) > 1. \end{cases}$$

Furthermore

$$\frac{1}{N}\#\Big\{v \in \mathcal{L}_1\Big(G^N(p^N,\kappa)\Big) \,:\, \text{type}(v) = i\Big\} \xrightarrow{d} \pi_i \zeta_i^{\pi,\kappa}, \quad i \in [k].$$

Note. Bollobás, Janson and Riordan treat more general type-spaces in [15] and so determine criticality with different notation. In the language of k types, consider defining an inner product $\langle x, y\rangle_\pi := \sum_{i=1}^k x_i y_i \pi_i$. With respect to this inner product, $\kappa \circ \pi$ is self-adjoint, since κ is symmetric. If we extend this inner product to a norm $||\cdot||_{L^2(\pi)}$, it can be seen that the operator norm $||T_\kappa||_{L^2(\pi)}$ in [15] corresponds exactly to $\rho(\kappa \circ \pi)$ as used here.

To motivate the role of the principal eigenvectors in this setting when $\rho(\kappa \circ \pi) = 1 + \epsilon$, we linearise (3.2). We obtain

$$\zeta^{\pi,\kappa} = (\kappa \circ \pi)\zeta^{\pi,\kappa} - \Theta(||\zeta^{\pi,\kappa}||^2),$$

which is consistent with $\zeta^{\pi,\kappa} = \Theta(\epsilon)\nu(\kappa \circ \pi)$, where we recall that $\nu(A)$ is the *right* eigenvector corresponding to the Perron root of a positive matrix A. It is a consequence of the self-adjointness of $\kappa \circ \pi$ with respect to $\langle \cdot, \cdot \rangle_\pi$ that, when π is positive,

$$\frac{\mu_i(\kappa \circ \pi)}{\pi_i} \propto \nu_i(\kappa \circ \pi).$$

In particular, we expect

$$\mathbb{P}(|\Xi^{\pi,\kappa}| = \infty, \text{type(root)} = i) \propto \mu_i(\kappa \circ \pi),$$

with magnitude $\Theta(\epsilon)$ when $\rho(\kappa \circ \pi) = 1 + \epsilon$.

Returning to $G^N(p, \kappa)$, with $\pi = p/N$; heuristically, whenever $\rho(\kappa \circ \pi) \leq 1 + \epsilon$, it is very unlikely that the largest component of $G^N(p, \kappa)$ will be substantially larger than ϵN. Furthermore, the proportion of types in any large component will be close to $\mu(\kappa \circ \pi)$. In the rest of this chapter, we prove concrete results of this type, which we will use in the next chapter when we return to frozen percolation.

3.2 Exponential bounds on component size

In this section, we consider IRGs $G^N(p,\kappa)$ satisfying $\rho(\kappa\circ p/N)\le 1+\epsilon$. We show exponential upper tail bounds for the size of the largest component in such a graph, uniformly as $N\to\infty$. Precisely, we will show the following result.

THEOREM 3.9. Fix $0<\eta<\infty$ and then $\epsilon\in(0,\frac{\eta^4}{8}\wedge\frac{1}{2})$. Then there exist $N_0=N_0(\epsilon,\eta)\in\mathbb{N}$ and constants $\chi=\chi(\epsilon,\eta)<\infty$ and $\Gamma=\Gamma(\epsilon,\eta)>0$, such that for any $N\ge N_0$ and given

- a kernel $\kappa\in[\eta,\infty)^{k\times k}$;
- a vector $p\in\mathbb{N}^k$ such that $\sum p_i=N$ and $p_i/N\ge\eta$ for each i;
- the eigenvalue condition $\rho(\kappa\circ p/N)\le 1+\epsilon$ is satisfied;

the following holds. For $|C(v)|$ the size of the component of a uniformly-chosen vertex in $G^N(p,\kappa)$,

$$\mathbb{P}(|C(v)|\ge\chi N)\le\exp(-N\Gamma).$$

As $\epsilon\to 0$ with η fixed, we may choose the constants such that $\chi(\epsilon,\eta)\to 0$.

Corollary 3.10. If $L_1(G)$ is the size of the largest component in a graph G satisfying these same conditions, then

$$\mathbb{P}\Big(L_1(G^N(p,\kappa))\ge\chi N\Big)\le\exp(-N\Gamma),$$

for large enough N.

The proof of Theorem 3.9 occupies the rest of this section.

Note. As Theorem 1.4 of [33], Janson and Riordan show exponential bounds in probability for the size of the largest component when κ is fixed and p/N converges to a fixed $\pi\in\Pi_1$. An argument along the lines of Lemma 3.12 below can be used to lift this statement to the uniformity we require here.

We will present a different proof, using a multitype exploration process that is, to the best of our knowledge, novel.

Note. In the setting of the original Erdős–Rényi model, O'Connell [56] proves a stronger result, namely that the size of the largest component of $G(N, c/N)$ satisfies a large deviation principle with rate N, and positive rate function away from ζ_c. The argument involves a careful direct calculation. Such an argument is hard to reproduce in the multitype setting because the probability of forming a component depends on the number of edges between vertices of each pair of types.

3.2.1 Technical preliminaries - positive matrices and eigenvectors

We summarise the results we will require about positive matrices here. Their proofs are given in Section 3.4.

THEOREM (Collatz–Wielandt formula [17, 70])**.** Given $A \in \mathbb{R}_{\geq 0}^{k\times k}$,
let $f(x) := \min_{\substack{1\leq i\leq n\\ x_i\neq 0}} \frac{[xA]_i}{x_i}$. Then

$$\rho(A) = \max_{x\in\mathbb{R}^k_{\geq 0}\setminus\{0\}} f(x). \tag{3.3}$$

Corollary 3.11. The Perron root ρ is non-decreasing as a function of $\mathbb{R}_+^{k\times k}$ and satisfies

$$\min_{i\in[k]} \sum_{j=1}^{k} A_{i,j} \leq \rho(A) \leq \max_{i\in[k]} \sum_{j=1}^{k} A_{i,j}. \tag{3.4}$$

In Chapter 5, we will need to consider limits uniformly among all measure-kernel pairs with a given Perron root. The following compactification lemma shows that sometimes we may reduce the problem to checking a finite number of measure-kernel pairs with a different Perron root.

Lemma 3.12. For any $0 < \bar{\Lambda} < \Lambda$, and $K < \infty$ there exist $M \in \mathbb{N}$, and $\pi^{(1)}, \dots, \pi^{(M)} \in \Pi_{\leq 1}$ and kernels $\kappa^{(1)}, \dots, \kappa^{(M)} \in \mathbb{R}_{\geq 0}^{k\times k}$ such that

- $\rho(\kappa^{(m)} \circ \pi^{(m)}) = \bar{\Lambda}$ for each $m \in [M]$,
- for any $\pi \in \Pi_{\leq 1}$ and kernel $\kappa \in [0, K]^{k\times k}$ with $\rho(\kappa \circ \pi) \geq \Lambda$, there is some $m \in [M]$ for which $\pi^{(m)} \leq \pi$ and $\kappa^{(m)} \leq \kappa$. (Recall that for both vectors and matrices, the ordering $\leq$ is taken coordinate-wise.)

Remark. The condition $\kappa_{\max} \leq K$ is necessary. Otherwise, consider

$$\kappa_{i,j} = \begin{cases} L & i = j = 1 \\ \frac{1}{L} & \text{otherwise}, \end{cases} \qquad \pi = \left(\frac{\Lambda}{L}, \ldots, \frac{\Lambda}{L}\right),$$

and allow $L \to \infty$.

Recall that $\mu(A), \nu(A)$ are the principal left- and right-eigenvectors respectively of a positive matrix A, normalised such that $\sum_{i=1}^{k} \mu_i(A) = \sum_{i=1}^{k} \nu_i(A) = 1$. We will also work with $\Pi_{\leq 1} \cap [\eta, 1]^k$, the set of sub-distributions where every component is at least η.

Proposition 3.13. Fix $0 < \eta < T < \infty$. Then,

$$\lim_{R \to \infty} \sup_{\substack{\pi \in \Pi_{\leq 1} \cap [\eta,1]^k \\ \kappa \in [\eta, T]^{k \times k}}} \sup_{v \in \Pi_1} \left|\left| \frac{v(\kappa \circ \pi)^R}{||v(\kappa \circ \pi)^R||_1} - \mu(\kappa \circ \pi) \right|\right|_1 = 0. \tag{3.5}$$

Remark. Since $\frac{v(\kappa \circ \pi)^R}{||v(\kappa \circ \pi)^R||_1}$ is invariant under non-zero scalar multiplication of v, we could alternatively take a supremum over $v \in \mathbb{R}^k_{\geq 0} \backslash \{0\}$, in the statement (3.5).

Remark. The non-uniform version of (3.5) is due to Perron [57], and a related limiting matrix is often called the *Perron projection.* Related results appear in the multitype branching process literature, including [36], for which [9] offers a comprehensive summary.

We will require a stronger version of Proposition 3.13, for large products of matrices close to a fixed matrix. For $A \in \mathbb{R}_+^{k \times k}$, and $\theta > 0$, define

$$\mathbb{B}_\theta(A) := \{B \in \mathbb{R}_+^{k \times k} \,:\, |B_{i,j} - A_{i,j}| \leq \theta, \forall i, j \in [k]\}, \tag{3.6}$$

the set of positive kernels whose entries differ from those of A by at most θ.

Lemma 3.14. For all $0 < \eta < T < \infty$ with $\eta < 1$, and $\delta > 0$, there exist $\theta = \theta(\delta, \eta, T) \in (0, \eta^2)$ and $R = R(\delta, \eta, T) < \infty$ such that

$$\left|\left| \frac{vD^{(1)} \ldots D^{(R)}}{||vD^{(1)} \ldots D^{(R)}||_1} - \mu(\kappa \circ \pi) \right|\right|_1 < \delta, \tag{3.7}$$

for all $v \in \mathbb{R}^k_{\geq 0} \backslash \{0\}$, $\kappa \in [\eta, T]^{k \times k}$, $\pi \in \Pi_{\leq 1} \cap [\eta, 1]^k$, and $D^{(1)}, \ldots, D^{(R)} \in \mathbb{B}_\theta(\kappa \circ \pi)$.

In later chapters, we require a local Lipschitz result for μ.

Lemma 3.15. Let $\mathbb{A}$ be a compact subset of $\mathbb{R}_{\geq 0}^{k\times k}$ with the property that for any $A \in \mathbb{A}$, the Perron root $\rho(A)$ is a simple eigenvalue. Then there exists a constant $C(\mathbb{A}) < \infty$ such that, for all matrices $A, A' \in \mathbb{A}$,

$$||\mu(A) - \mu(A')||_1 \leq C(\mathbb{A}) \max_{i,j\in[k]} |A_{i,j} - A'_{i,j}|. \tag{3.8}$$

In particular, for any $0 < \eta < T < \infty$, there exists $C(\eta, T) < \infty$ such that, for all matrices $A, A' \in [\eta, T]^{k\times k}$,

$$||\mu(A) - \mu(A')||_1 \leq C(\eta, T) \max_{i,j\in[k]} |A_{i,j} - A'_{i,j}|. \tag{3.9}$$

3.2.2 A multitype exploration process

In Section 1.2.1, we saw how an exploration process could be used to encode the sizes of components in a graph. This is particularly useful in $G(N, p)$, since the exploration process is Markov, and its increments can be characterised with binomial distributions, as in (1.17).

We will encode the sizes and type counts of components in an inhomogeneous random graph via a similar exploration process. In this setting, we must also track the types of the vertices as we explore. Thus our multitype exploration process will be $\mathbb{Z}^k$-valued. In Section 1.2.1, we discussed classical orderings, for example when the next vertex for exploration is chosen in a depth-first manner. This enables us to recover the genealogy of the component. The calculations which follow are valid for any (non-look-ahead) ordering, but it is most natural in the multitype setting to choose the next vertex for exploration *uniformly at random* from the candidate vertices at each stage, as the resulting process will then be Markov. For readability, we recall the full definition.

Definition 3.16. Given a graph G with k types, we define the following *multitype exploration process*. We choose v_1 uniformly from $V(G)$, and set $\mathcal{Z}^0 := \{v_1\}$. Then, for each $m \geq 1$ in turn, we choose v_{m+1} uniformly from $\mathcal{Z}^m := \Gamma(v_1) \cup \ldots \cup \Gamma(v_m) \backslash \{v_1, \ldots, v_m\}$, unless $\mathcal{Z}^m = \varnothing$, in which case we choose v_{m+1} uniformly from $V(G)\backslash\{v_1, \ldots, v_m\}$.

We now define $Z^m, R^m \in \mathbb{Z}_{\geq 0}^k$ to be the counts of types in $\mathcal{Z}^m$ and $\{v_1, \dots, v_m\}$ respectively. That is:

$$Z_i^m := \#\{v \in \mathcal{Z}^m : \text{type}(v) = i\}, \quad m \geq 0,\, i \in [k],$$

$$R_i^m := \#\{v \in \{v_1, \dots, v_m\} : \text{type}(v) = i\}, \quad m \geq 0,\, i \in [k].$$

In particular, we know that the size of the component containing v_1 is given by

$$|C(v_1)| = \tau(Z) := \inf\{m \geq 1 : Z^m = 0\}.$$

We now return to the case where the graph is distributed as $G^N(p, \kappa)$. Then $(Z^m, R^m)_{m \geq 0}$ is Markov since we can define the transitions as follows for any $m \in \{0, 1, \dots, N-1\}$.

- If $Z^m = 0$, then $\mathcal{Z}^m = \varnothing$, and so v_{m+1} is chosen uniformly from the remaining vertices, independently of the history of the process. Conditional on R^m, for each $i \in [k]$, with probability $\frac{p_i - R_i^m}{N-m}$, v_{m+1} has type i, and on this event,

$$R^{m+1} = R^m + \mathbf{e}^{(i)}, \quad Z_j^{m+1} \stackrel{d}{=} \text{Bin}\Big(p_j - R_j^{m+1}, 1 - e^{-\kappa_{i,j}/N}\Big), \quad j \in [k], \tag{3.10}$$

  where $\mathbf{e}^{(i)}$ is the ith unit vector.

- If $Z^m \neq 0$, then v_{m+1} is chosen uniformly from $\mathcal{Z}^m$. Conditional on (Z^m, R^m), for each $i \in [k]$, with probability $\frac{Z_i^m}{||Z^m||_1}$, v_{m+1} has type i, and, on this event,

$$R^{m+1} = R^m + \mathbf{e}^{(i)},$$

$$Z_j^{m+1} - Z_j^m + \delta_{i,j} \stackrel{d}{=} \text{Bin}\Big(p_j - R_j^m - Z_j^m, 1 - e^{-\kappa_{i,j}/N}\Big), \quad j \in [k]. \tag{3.11}$$

Comparison with the multitype branching process

First we need to define an alternative version of the multitype branching process which is never empty.

Definition 3.17. Given $\kappa \in \mathbb{R}_{\geq 0}^{k \times k}$ and $\pi \in \Pi_{\leq 1} \backslash \{0\}$, we define $\bar{\Xi}^{\pi,\kappa}$ to be a random multitype tree with the same distribution as $\Xi^{\pi,\kappa}$ conditional on $\varnothing \in \Xi^{\pi,\kappa}$. $\varnothing$ is the root of the Ulam–Harris tree $\mathcal{U}$, which supports $\Xi^{\pi,\kappa}$. This condition is equivalent to $\Xi^{\pi,\kappa} \neq \varnothing$. Then $\bar{\Xi}^{\pi,\kappa}$ can be constructed by taking the type of the root to be given by distribution $\frac{\pi}{||\pi||_1}$, and thereafter using the same offspring distributions as for $\Xi^{\pi,\kappa}$. It follows immediately that when $\pi \in \Pi_1$, the distributions of $\Xi^{\pi,\kappa}$ and $\bar{\Xi}^{\pi,\kappa}$ coincide.

In Chapter 5, we will want to compare the size of the component of a uniformly-chosen vertex v in $G^N(p,\kappa)$, and $|\bar{\Xi}^{p/N,\kappa}|$, the size of the corresponding multitype branching process. The following result couples these two objects via their multitype exploration processes. Note that this approach is not original. For example, Bollobás, Janson and Riordan use a similar approach in their proof of Lemma 9.6 [15], where instead they compare to $|\bar{\Xi}^{p/N,(1+\epsilon)\kappa}|$.

Proposition 3.18. Let $N \in \mathbb{N}$, $p \in \mathbb{N}_0^k$, and $\kappa \in \mathbb{R}_{\geq 0}^{k \times k}$, and let $|C(v)|$ be the size of the component of a uniformly-chosen vertex in $G^N(p,\kappa)$. Then

$$|C(v)| \leq_{st} |\bar{\Xi}^{p/N,\kappa}|. \tag{3.12}$$

Proof. For brevity, we write $\pi = p/N$ and $\bar{\Xi}$ for $\bar{\Xi}^{\pi,\kappa}$ during this proof. We show that there exists a coupling of $C(v)$ and its exploration process $(Z^m)_{m \geq 0}$ with $\bar{\Xi}$ and an exploration process $(\bar{Z}^m)_{m \geq 0}$ of $\bar{\Xi}$, such that $\bar{Z}^m \geq Z^m$ for $m \leq \tau(Z)$. We prove this by induction on $m \geq 0$.

Clearly we can couple $C(v)$ and $\bar{\Xi}$ such that the type of v and the type of the root of $\bar{\Xi}$ are the same. (Note that, unlike Ξ, $\bar{\Xi}$ has a root with probability 1.) Then $\bar{Z}_j^1$ is the number of children of the root of $\bar{\Xi}$ with type j. If v and the root have type i, then $\bar{Z}_j^1 \overset{d}{=} \text{Po}(\kappa_{i,j}\pi_j)$. By comparing the probability that each is equal to zero, it is easily seen that

$$\bar{Z}_j^1 \geq_{st} \text{Bin}\Big(p_j, 1 - e^{-\kappa_{i,j}/N}\Big) \overset{(3.10)}{\geq_{st}} Z_j^1.$$

Therefore, $\bar{Z}^1 \geq_{st} Z^1$.

Suppose that $v_1, \dots, v_m$ and $\bar{v}_1, \dots, \bar{v}_m$ are the initial vertices in the explorations of $C(v)$ and $\bar{\Xi}$ respectively, and assume that $\bar{Z}^m \geq Z^m$. Then if v_{m+1} is chosen with type i, there exists a $\bar{v} \in \Gamma(\bar{v}_1) \cup \dots \cup \Gamma(\bar{v}_m) \backslash \{\bar{v}_1, \dots, \bar{v}_m\}$ in $\bar{\Xi}$ with type i by assumption, so take this to be $\bar{v}_{m+1}$. Then

$$\bar{Z}_j^{m+1} - \bar{Z}_j^m + \delta_{i,j} \stackrel{d}{=} \mathrm{Po}(\kappa_{i,j}\pi_j).$$

This stochastically dominates $\mathrm{Bin}\Big(p_j, 1 - e^{-\kappa_{i,j}/N}\Big)$, and thus from (3.11)

$$\bar{Z}^{m+1} - \bar{Z}^m \geq_{st} Z^{m+1} - Z^m.$$

So by induction we obtain $\bar{Z}^m \geq Z^m$ for $m \leq \tau(Z)$, and thus $\tau(\bar{Z}) \geq \tau(Z)$, and the result follows. □

Remark. If instead we take v to be chosen uniformly from $[N]$, rather than from $V(G^N(p, \kappa)) = [\sum p_i]$, then (3.12) would hold with Ξ instead of $\bar{\Xi}$, with an almost-identical argument.

A remark about limits of Z

Given $\pi \in \Pi_1$, and $\kappa \in \mathbb{R}_{\geq 0}^{k \times k}$, consider the following coupled differential equations for $\mathbb{R}^k$-valued processes $((r(t), z(t)), t \geq 0)$:

$$r(0) = z(0) = 0,$$

$$\dot{r}(0) = \mu(\kappa \circ \pi), \quad \dot{z}(0) = (1 - \rho(\kappa \circ \pi))\mu(\kappa \circ \pi),$$

$$\dot{r}(t) = \frac{z(t)}{||z(t)||_1}, \quad \dot{z}(t) = -\frac{z(t)}{||z(t)||_1} + \frac{z(t)}{||z(t)||_1}(\kappa \circ (\pi - r(t) - z(t))), \quad t > 0. \quad (3.13)$$

When $\rho(\kappa \circ \pi) > 1$, we conjecture that there exists $T = T(\kappa, \pi) > 0$, such that there exists a unique solution to (3.13) on $[0, T)$, such that

$$z(t) > 0, \quad t \in (0, t), \quad \lim_{t \to T} z(t) = 0.$$

When the size of the graph N is large, and $p/N \approx \pi$, we expect the rescaled multitype exploration process $(\frac{1}{N} Z^{\lfloor tN \rfloor}, t \in [0,T])$ to be well-approximated by $(z(t), t \in [0,T])$.

We will not prove an exact convergence result of this type here, which would lead to an alternative expression for the size and type-measure of the largest component in a supercritical IRG.

3.2.3 Proof of Theorem 3.9

We will study the multitype exploration process separately on two time intervals. In the first regime, the process is growing, and we find upper bounds on how far it can grow. In the second regime, irrespective of what has happened during the first time interval, we can bound the process above by a related decreasing process. In both cases, since the process is vector-valued, we will consider the projections of Z in a fixed direction, which will be given by a principal eigenvector. We reduce the problem to a bound on the probability that a large sum of 'almost IID' random variables substantially exceeds its mean.

We start with the following lemma, which gives uniform control on the proportion of vertices (across all types) that need to be removed from an inhomogeneous random graph with eigenvalue $(1+\epsilon)$ to give a graph with eigenvalue at most $(1-\epsilon)$. We will use this to divide our analysis into two regimes.

Lemma 3.19. Fix $\eta > 0$, and set $\theta(\eta) := \frac{\eta^4}{8}$. Then there exists a function $c : (0, \theta(\eta)) \to (0, \eta/2]$ such that:

- for all $\epsilon \in (0, \theta(\eta))$, whenever we take $\pi \in \Pi_{\leq 1}$ satisfying $\pi_i \geq \eta$ for all $i \in [k]$, and $\kappa \in [\eta, \infty)^{k \times k}$ such that $\rho(\kappa \circ \pi) \leq 1 + \epsilon$, then for any $\pi' \in \Pi_{\leq 1}$ satisfying

  $$\pi' \leq \pi, \quad \text{and} \quad ||\pi - \pi'||_1 \geq c(\epsilon),$$

  we have $\rho(\kappa \circ \pi') \leq 1 - \epsilon$;
- as $\epsilon \to 0$, $c(\epsilon) \to 0$.

Remark. The proof, which is postponed until Section 3.4, shows that $c(\epsilon) = \frac{4\epsilon}{\eta^3}$ satisfies these conditions.

Proof of Theorem 3.9. Within this proof, we take $\pi := \frac{p}{N}$. The proof proceeds by first bounding $||Z^{\lceil c(\epsilon)N\rceil}||_1$ exponentially in probability, where $c(\epsilon)$ is as given by Lemma 3.19. We achieve this by considering the projection onto the principal right-eigenvector of $\kappa \circ \pi$. Thereafter, the evolution is controlled by matrices with eigenvalue at most $1-\epsilon$. We bound this process above, stochastically, at least until it first hits zero, from which we will obtain our estimate on the component size, using a similar projection-onto-eigenvector decomposition.

During the first half of this proof, for brevity we write ρ for $\rho(\kappa \circ \pi)$, and ν for the right-eigenvector $\nu(\kappa \circ \pi)$. Now, let (Z^m, R^m) be the multitype exploration process of $G^N(p,\kappa)$ as defined in Section 3.2.2, and let $\mathcal{F} = (\mathcal{F}^m)$ be its natural filtration. For $m \geq 0$, define $S^m := Z^m \cdot \nu$ to be the magnitude of the projection of the $\mathbb{N}_0^k$-valued exploration process onto the span of this eigenvector. Then, when it is not at zero, the evolution of S^m follows from (3.11). For each $i \in [k]$, let $\mathcal{A}_i^m$ be the event that v_{m+1} has type i, which, conditional on $\mathcal{F}^m$, occurs with probability $\frac{Z_i^m}{||Z^m||_1}$, whenever Z^m is not zero.

We will slightly abuse notation by writing $\mathbb{E}\left[S^{m+1} - S^m \,\middle|\, \mathcal{F}^m, \mathcal{A}_i^m\right]$ for

$$\frac{\mathbb{E}\Big[(S^{m+1} - S^m)\mathbb{1}_{\mathcal{A}_i^m} \,\Big|\, \mathcal{F}^m\Big]}{\frac{Z_i^m}{||Z^m||_1}}.$$

That is, conditioning on the sigma-algebra $\mathcal{F}^m$, *and* the event $\mathcal{A}_i^m$.

As motivation, we consider the conditional expected increments of (S^m). For each $i \in [k]$, from (3.10) and (3.11),

$$\begin{aligned}
\mathbb{E}\Big[S^{m+1} - S^m \,\Big|\, \mathcal{F}^m, \mathcal{A}_m^i\Big] &\leq -\nu_i \mathbb{1}_{\{Z^m \neq 0\}} + \sum_{j\in[k]} \nu_j p_j (1 - e^{-\kappa_{i,j}/N}) \\
&\leq -\nu_i \mathbb{1}_{\{Z^m \neq 0\}} + \sum_{j\in[k]} \kappa_{i,j}\pi_j \nu_j = \nu_i(\rho - \mathbb{1}_{\{Z^m \neq 0\}}) \\
&\leq \rho - \mathbb{1}_{\{Z^m \neq 0\}} \leq \epsilon + \mathbb{1}_{\{Z^m = 0\}}. \qquad (3.14)
\end{aligned}$$

We will shortly use a Chernoff bound, so we need to estimate the conditional MGF of the increments as well. First, we choose $\lambda > 0$ small enough that $e^{\lambda} - 1 \le (1 + \frac{\epsilon}{2})\lambda$. By convexity, it follows at once that

$$e^{\lambda\alpha} - 1 \le (1 + \tfrac{\epsilon}{2})\lambda\alpha, \quad \forall \alpha \in [0, 1]. \tag{3.15}$$

We now bound the MGFs of the increments of S^m. Since we are only interested in the exploration process until the first time it hits zero, we do not include the case (3.10) in the subsequent calculation. To this end, we let $\tau(Z) := \min\{m : Z^m = 0\}$. Naturally, $\tau(Z)$ is an $\mathcal{F}$-stopping time. We consider the process $\left(S^{m\wedge\tau(Z)}\right)_{m\ge 0}$, that is S absorbed at zero. Then,

$$\begin{aligned}
&\mathbb{E}\Big[\exp(\lambda(S^{(m+1)\wedge\tau(Z)} - S^{m\wedge\tau(Z)})) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big] \\
&\quad\le \mathbb{1}_{\{\tau(Z)\le m\}} + \mathbb{1}_{\{\tau(Z)>m\}} e^{-\lambda\nu_i} \prod_{j\in[k]} \Big[1 + (1 - e^{-\kappa_{i,j}/N})(e^{\lambda\nu_j} - 1)\Big]^{p_j} \\
&\quad\le \mathbb{1}_{\{\tau(Z)\le m\}} + \mathbb{1}_{\{\tau(Z)>m\}} e^{-\lambda\nu_i} \prod_{j\in[k]} \big[1 + (1 + \tfrac{\epsilon}{2}) \cdot \tfrac{\kappa_{i,j}}{N}\lambda\nu_j\big]^{p_j},
\end{aligned} \tag{3.16}$$

using (3.15) and the bound $1 - \exp(-x) \le x$. From this, using $\log(1 + x) \le x$,

$$\begin{aligned}
&\log\Big(\mathbb{E}\Big[\exp\Big(\lambda(S^{(m+1)\wedge\tau(Z)} - S^{m\wedge\tau(Z)})\Big) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big]\Big) \\
&\quad\le \mathbb{1}_{\{\tau(Z)>m\}} \left[-\lambda\nu_i + \sum_{j\in[k]} p_j(1 + \tfrac{\epsilon}{2}) \cdot \tfrac{\kappa_{i,j}}{N}\lambda\nu_j\right] \\
&\quad\le \mathbb{1}_{\{\tau(Z)>m\}} \left[-\lambda\nu_i + \sum_{j\in[k]} \kappa_{i,j}\pi_j(1 + \tfrac{\epsilon}{2})\lambda\nu_j\right] \\
&\quad\le \mathbb{1}_{\{\tau(Z)>m\}} \lambda\nu_i(-1 + (1 + \tfrac{\epsilon}{2})\rho) \\
&\quad\le \lambda(-1 + (1 + \tfrac{\epsilon}{2})(1 + \epsilon)),
\end{aligned} \tag{3.17}$$

since $\nu_i \le 1$. Now the RHS of (3.17) has no dependence on i. After iterated application of the tower law, we obtain

$$\log\Big(\mathbb{E}\Big[\exp\Big(\lambda(S^{m\wedge\tau(Z)} - S^0)\Big)\Big]\Big) \le \lambda m(-1 + (1 + \tfrac{\epsilon}{2})(1 + \epsilon)).$$

Then, applying Markov's inequality in the usual way, we find

$$\begin{aligned}\log \mathbb{P}(S^m \geq 2\epsilon m, \tau(Z) \geq m) &\leq \log \mathbb{P}\Big(S^{m\wedge\tau(Z)} - S^0 \geq 2\epsilon m - 1\Big)\\ &\leq \log \mathbb{E}\Big[\exp\Big(\lambda\Big(S^{m\wedge\tau(Z)} - S^0\Big)\Big)\Big] - \lambda(2\epsilon m - 1)\\ &\leq \lambda m\big[-1 + (1+\tfrac{\epsilon}{2})(1+\epsilon) - 2\epsilon\big] + \lambda. \qquad (3.18)\end{aligned}$$

Recall the definition of $c(\epsilon)$ from Lemma 3.19. Observe that $\lambda\big(2\epsilon + 1 - (1+\frac{\epsilon}{2})(1+\epsilon)\big) > 0$ since we demanded $\epsilon < \frac{1}{2}$, and λ depends only on ϵ. So take any $0 < \gamma = \gamma(\epsilon) < \lambda\big(2\epsilon + 1 - (1+\frac{\epsilon}{2})(1+\epsilon)\big)$. Then there exists $N_0 = N_0(\epsilon) \in \mathbb{N}$ such that, taking $m = \lceil c(\epsilon)N\rceil$ in (3.18), we have for all $N \geq N_0$,

$$\mathbb{P}\Big(S^{\lceil c(\epsilon)N\rceil} \geq 2\epsilon c(\epsilon)N,\, \tau(Z) \geq c(\epsilon)N\Big) \leq \exp(-c(\epsilon)\gamma N). \qquad (3.19)$$

Finally, we convert this into a probabilistic statement about $||Z^{\lceil c(\epsilon)N\rceil}||_1$. Observe that

$$\nu_i = \frac{1}{\rho(\kappa\circ\pi)}\sum_{j=1}^{k}\kappa_{i,j}\pi_j\nu_j \geq \frac{1}{1+\epsilon}\min_{j\in[k]}\kappa_{i,j}\pi_j \geq \frac{\eta^2}{1+\epsilon} \geq \frac{\eta^2}{2}. \qquad (3.20)$$

Then, for any $m \geq 0$, $||Z^m||_1 \leq S^m/\min_i \nu_i$, and so on the event $\{\tau(Z) \geq m\}$, $||Z^m||_1 \leq S^{m\wedge\tau(Z)}/\min_i \nu_i$. So, from (3.19), we conclude

$$\mathbb{P}\Big(||Z^{\lceil c(\epsilon)N\rceil}||_1 \geq 4\epsilon\eta^{-2}c(\epsilon)N,\, \tau(Z) \geq c(\epsilon)N\Big) \leq \exp(-c(\epsilon)\gamma N), \qquad (3.21)$$

for $N \geq N_0$.

Now we consider the evolution beyond time $c(\epsilon)N$. We define

$$p_i' = p_i - R_i^{\lceil c(\epsilon)N\rceil}, \quad \pi_i' = p_i'/N, \quad i \in [k].$$

Since $||R^{\lceil c(\epsilon)N\rceil}||_1 = \lceil c(\epsilon)N\rceil$, we have $||\pi - \pi'|| \geq c(\epsilon)$. Using this, and the fact that $\rho(\kappa\circ\pi) \leq 1+\epsilon$, we know from Lemma 3.19 that $\rho' := \rho(\kappa\circ\pi') \leq 1-\epsilon$ almost surely. Recall that $\pi_i \geq \eta \geq 2c(\epsilon)$, and so to avoid errors from the ceiling function, we assume throughout that N is large enough that $\pi'_i \geq \eta/3$, for all $i \in [k]$. We may consider ν', the principal right-eigenvector of $\kappa \circ \pi'$, normalised so that $||\nu'||_1 = 1$.

For $m \geq \lceil c(\epsilon)N \rceil$, define $S'^m := Z^m \cdot \nu'$ to be the magnitude of the projection of Z onto this (random) eigenvector ν'. As before, S'^m is adapted to $\mathcal{F}$ and, when it is not at zero, the distribution of the increments of S'^m follows directly from (3.11). Since ν' is $\mathcal{F}^{\lceil c(\epsilon)N \rceil}$-measurable, we can repeat the calculation (3.16), for $m \geq \lceil c(\epsilon)N \rceil$.

$$\mathbb{E}\Big[\exp(\lambda(S'^{(m+1)} - S'^m)) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big]$$
$$\leq \mathbb{1}_{\{\tau(Z) \leq m\}} + \mathbb{1}_{\{\tau(Z) > m\}} e^{-\lambda \nu'_i} \prod_{j \in [k]} \Big[1 + (1 - e^{-\kappa_{i,j}/N})(e^{\lambda \nu'_j} - 1)\Big]^{p'_j}.$$

Then,

$$\log\Big(\mathbb{E}\Big[\exp(\lambda(S'^{(m+1)} - S^m)) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big]\Big) \leq \mathbb{1}_{\{\tau(Z) > m\}} \lambda \nu'_i (-1 + (1 + \tfrac{\epsilon}{2})\rho').$$

By the same argument as (3.20), since $\rho' \leq 1 - \epsilon$ and $\pi'_i \geq \eta/3$ for all $i \in [k]$, we obtain $\nu'_i \geq \eta^2/3$. Recall also that $\epsilon \in (0, \frac{1}{2})$, and so

$$(1 + \tfrac{\epsilon}{2})\rho' \leq (1 + \tfrac{\epsilon}{2})(1 - \epsilon) < 1 - \frac{\epsilon}{2}.$$

Therefore,

$$\log\Big(\mathbb{E}\Big[\exp(\lambda(S'^{m+1} - S'^m)) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big]\Big) \leq -\lambda \mathbb{1}_{\{\tau(Z) > m\}} \frac{\epsilon \eta^2}{6}. \tag{3.22}$$

With the aim of avoiding the requirement to carry this indicator function through the tower law, we define $\tau'(Z) = \tau(Z) \wedge \lceil c(\epsilon)N \rceil$, and then for $m \geq \lceil c(\epsilon)N \rceil$,

$$\bar{S}^m := S'^m - \frac{\epsilon \eta^2}{6}\Big((m - \tau'(Z)) \vee 0\Big). \tag{3.23}$$

That is, $\bar{S}^{\lceil c(\epsilon)N \rceil} = S'^{\lceil c(\epsilon)N \rceil}$, and then $\bar{S}$ follows S' if Z has not yet hit zero. But if Z has already hit zero, or when it hits zero after $\lceil c(\epsilon)N \rceil$, the increments of $\bar{S}$ are deterministic and fixed (and negative). We can then rewrite (3.22) in terms of $\bar{S}^m$ as

$$\log\Big(\mathbb{E}\Big[\exp(\lambda(\bar{S}^{m+1} - \bar{S}^m)) \,|\, \mathcal{F}^m, \mathcal{A}_i^m\Big]\Big) \leq -\lambda\frac{\epsilon\eta^2}{6}. \tag{3.24}$$

Since the RHS of (3.24) has no dependence on i nor $\mathcal{F}^m$, we apply the tower law to obtain

$$\log\Big(\mathbb{E}\Big[\exp(\lambda(\bar{S}^{\lceil c(\epsilon)N\rceil+m} - \bar{S}^{\lceil c(\epsilon)N\rceil})) \,|\, \mathcal{F}^{\lceil c(\epsilon)N\rceil}\Big]\Big) \leq -\lambda m\frac{\epsilon\eta^2}{6}. \tag{3.25}$$

Then, taking $m = 36\eta^{-4}\lceil c(\epsilon)N\rceil$, and applying Markov's inequality again, we find

$$\begin{aligned}
&\log\mathbb{P}\Big(\bar{S}^{\lceil c(\epsilon)N\rceil+36\eta^{-4}\lceil c(\epsilon)N\rceil} - \bar{S}^{\lceil c(\epsilon)N\rceil} \geq -4\epsilon\eta^{-2}c(\epsilon)N\Big)\\
&\quad\leq -\lambda\cdot 36\eta^{-4}c(\epsilon)N\cdot\frac{\epsilon\eta^2}{6} + \lambda\cdot 4\epsilon\eta^{-2}c(\epsilon)N\\
&\quad\leq -2\lambda\epsilon\eta^{-2}c(\epsilon)N.
\end{aligned}$$

Finally, by patching together $S^{m\wedge\tau(Z)}$ and $\bar{S}^m$, we can now address the probability that $\tau(Z)$ is large.

$$\begin{aligned}
\mathbb{P}\Big(\tau(Z) \geq \lceil c(\epsilon)N\rceil + 36\eta^{-4}\lceil c(\epsilon)N\rceil\Big) &\leq \mathbb{P}\Big(\bar{S}^{\lceil c(\epsilon)N\rceil+36\eta^{-4}\lceil c(\epsilon)N\rceil} \geq 0, \tau(Z) \geq c(\epsilon)N\Big)\\
&\leq \mathbb{P}\Big(\bar{S}^{\lceil c(\epsilon)N\rceil} \geq 4\epsilon\eta^{-2}c(\epsilon)N, \tau(Z) \geq c(\epsilon)N\Big)\\
&\qquad + \mathbb{P}\Big(\bar{S}^{\lceil c(\epsilon)N\rceil+36\eta^{-4}\lceil c(\epsilon)N\rceil} - \bar{S}^{\lceil c(\epsilon)N\rceil} \geq -4\epsilon\eta^{-2}c(\epsilon)N\Big)\\
&\leq \mathbb{P}\Big(||Z^{\lceil c(\epsilon)N\rceil}||_1 \geq 4\epsilon\eta^{-2}c(\epsilon)N, \tau(Z) \geq c(\epsilon)N\Big)\\
&\qquad + \mathbb{P}\Big(\bar{S}^{\lceil c(\epsilon)N\rceil+36\eta^{-4}\lceil c(\epsilon)N\rceil} - \bar{S}^{\lceil c(\epsilon)N\rceil} \geq -4\epsilon\eta^{-2}c(\epsilon)N\Big)\\
&\overset{(3.19)}{\leq} \exp(-c(\epsilon)\gamma N) + \exp\Big(-2\lambda\epsilon\eta^{-2}c(\epsilon)N\Big),
\end{aligned} \tag{3.26}$$

for N large enough. We therefore take $\chi(\epsilon) := (1 + 36\eta^{-4})c(\epsilon)$, and observe that since $c(\epsilon) \to 0$ as $\epsilon \to 0$, we also have $\chi(\epsilon) \to 0$. Finally, choose positive $\Gamma < \min(c(\epsilon)\gamma, 2\lambda\epsilon\eta^{-2}c(\epsilon))$, and recalling that $\tau(Z) \overset{d}{=} |C(v_1)|$, the statement of Theorem 3.9 follows. □

3.3 Distribution of types in large components

For the purpose of the frozen percolation model to follow in Chapter 5, we require a result about the proportion of types in all large components; that is, not just the giant component (if it exists). We will approach this by considering the types of vertices connected at large distance from a uniformly-chosen vertex. For many choices of the root vertex there will be no vertices at large radius. But for large components, the majority of the vertices in such components will be a large distance from a uniformly-chosen vertex.

For any graph G with k types on N vertices, we will take v to be a uniformly-chosen vertex. Then, for $r = 0, 1, \dots, N-1$, define $W^r \in \mathbb{N}_0^k$ by,

$$W_i^r := \#\{\text{type } i \text{ vertices distance } r \text{ from } v\}, \quad i \in [k],$$

and $W_i^{\geq R} := \sum_{r=R}^{N-1} W_i^r$.

Remark. Throughout the literature, Z is often used to denote Galton–Watson and related processes. To avoid confusion, we reserve Z for reflected exploration processes in this thesis.

THEOREM 3.20. Fix constants $0 < \eta < T < \infty$. Now, for any $\delta > 0$, there exists $\epsilon = \epsilon(\delta, \eta, T) > 0$, and $R = R(\delta, \eta, T), N_0 = N_0(\delta, \eta, T) \in \mathbb{N}$ satisfying the following. Consider any $\kappa \in [\eta, T]^{k \times k}$ and $p \in \mathbb{N}_0^k$ satisfying $\sum p_i = N \geq N_0$ and $p_i \geq \eta N$, with $\rho(\kappa \circ \frac{p}{N}) \leq 1 + \epsilon$. Then $W^{\geq R}$ corresponding to v, a uniformly-chosen vertex in $G^N(p, \kappa)$ satisfies

$$\left|\left|\mathbb{E}\left[W^{\geq R}\right] - \mu(\kappa \circ \pi)||\mathbb{E}\left[W^{\geq R}\right]||_1\right|\right|_1 \leq \delta \left|\left|\mathbb{E}\left[W^{\geq R}\right]\right|\right|_1. \tag{3.27}$$

Furthermore, recall the definition of $\chi = \chi(\epsilon, \eta)$ from Theorem 3.9. Define the event $A_\chi := \{||W^{\geq R}||_1 \leq \chi N\}$, that the component containing v has size at most χN beyond radius R. Then we also have a constant $N_1 = N_1(\delta, \eta, T)$ such that whenever $N \geq N_1$,

$$\left|\left|\mathbb{E}\left[W^{\geq R}\mathbb{1}_{A_\chi}\right] - \mu(\kappa \circ \pi)||\mathbb{E}\left[W^{\geq R}\mathbb{1}_{A_\chi}\right]||_1\right|\right|_1 \leq \delta \left|\left|\mathbb{E}\left[W^{\geq R}\mathbb{1}_{A_\chi}\right]\right|\right|_1. \tag{3.28}$$

Remark. Neither the statements nor the proofs consider the value, or even the scale of $\left|\left|\mathbb{E}\left[W^{\geq R}\right]\right|\right|_1$. The results (3.27) and (3.28) deal only with the *direction* of $\mathbb{E}\left[W^{\geq R}\right]$, since they hold uniformly over $\rho \leq 1+\epsilon$, which includes subcritical, critical and supercritical regimes. That is, for $\rho = 1+\epsilon$, $\left|\left|\mathbb{E}\left[W^{\geq R}\right]\right|\right|_1 = \Theta(N)$ for any fixed R, whereas for fixed $\rho < 1$, $\left|\left|\mathbb{E}\left[W^{\geq R}\right]\right|\right|_1 = \Theta(1)$ as $N \to \infty$.

3.3.1 Convex combinations of matrix products

We begin with a further technical lemma. We show that convex combinations of matrices close to a fixed matrix can be written as a single matrix close to that fixed matrix. Such matrices will appear as expected multiplicative increments for a discrete-time process, and this lemma allows us the control the increments across multiple time-steps. We will also use Lemma 3.14 which asserts that applying a large enough product of matrices close to a fixed positive matrix to any vector has direction close to the principal eigenvector of the fixed matrix.

The combination of these results allows us to control the expected *proportion* of types at some large radius from a fixed vertex in an inhomogeneous random graph, irrespective of the expected *number* of vertices at this radius. Considering all distances from the fixed vertex simultaneously proves the required concentration result for the proportion of types in a typical large component, and all the estimates hold uniformly among graphs with bounded Perron root.

Recall the definition (3.6)

$$\mathbb{B}_\theta(A) := \{B \in \mathbb{R}_+^{k\times k} \,:\, |B_{i,j} - A_{i,j}| \leq \theta,\, \forall i,j \in [k]\},$$

of the set of positive kernels whose entries differ from those of A by at most θ.

Lemma 3.21. Given $A \in \mathbb{R}_+^{k\times k}$, $L \in \mathbb{N}$, and $\theta > 0$ such that $\min_{i,j} A_{i,j} > \theta$, consider non-negative non-zero vectors $x^{(1)}, \dots, x^{(L)} \in \mathbb{R}_{\geq 0}^k \backslash \{0\}$, and any L matrices

$$D^{(1)}, \dots, D^{(L)} \in \mathbb{B}_\theta(A),$$

and positive real numbers $p_1, \dots, p_L$ satisfying $\sum\limits_{\ell=1}^{L} p_\ell = 1$. Then there exists a matrix $\bar{D} \in \mathbb{B}_\theta(A)$ such that

$$p_1 x^{(1)} D^{(1)} + \ldots + p_L x^{(L)} D^{(L)} = (p_1 x^{(1)} + \ldots + p_L x^{(L)}) \bar{D}. \tag{3.29}$$

Proof. Let $\mathbf{1}$ be the $k \times k$ matrix where every entry is 1. Since the $x^{(i)}$s are non-negative, and each $D^{(l)} \in \mathbb{B}_\theta(A)$,

$$\begin{aligned} p_1 x^{(1)} D^{(1)} + \ldots + p_L x^{(L)} D^{(L)} &\leq p_1 x^{(1)} (A + \theta \mathbf{1}) + \ldots + p_L x^{(L)} (A + \theta \mathbf{1}) \\ &\leq (p_1 x^{(1)} + \ldots + p_L x^{(L)})(A + \theta \mathbf{1}). \end{aligned}$$

Similarly,

$$p_1 x^{(1)} D^{(1)} + \ldots + p_L x^{(L)} D^{(L)} \geq (p_1 x^{(1)} + \ldots + p_L x^{(L)})(A - \theta \mathbf{1}).$$

For ease of notation, set $y := p_1 x^{(1)} D^{(1)} + \ldots + p_L x^{(L)} D^{(L)}$ and $z := p_1 x^{(1)} + \ldots + p_L x^{(L)}$, so

$$z(A - \theta \mathbf{1}) \leq y \leq z(A + \theta \mathbf{1}). \tag{3.30}$$

Now, for each $j \in [k]$, set $c_j := \frac{y_j - [zA]_j}{||z||_1}$, so that $\sum_{i=1}^{k} z_i (A_{i,j} + c_j) = y_j$. Since the LHS is increasing in c_j, from (3.30) we have $|c_j| \leq \theta$.

So we may define $\bar{D} \in \mathbb{B}_\theta(A)$ via $\bar{D}_{i,j} = A_{i,j} + c_j$, and this satisfies (3.29). □

3.3.2 Proof of Theorem 3.20

Proof. We may insist $T > 2$, and first choose any $\epsilon \in (0, \frac{\eta^4}{8} \wedge \frac{1}{2})$ small enough that $\chi(\epsilon, \eta)$ as defined in Theorem 3.9 and $\theta(\delta, \eta, T)$ as defined in Lemma 3.14 satisfy

$$\chi(\epsilon, \eta) T^2 \leq \theta(\delta, \eta, T) < \eta^2. \tag{3.31}$$

(Recall we require this restriction on the range of ϵ to apply Theorem 3.9.)

Assume throughout that we have a graph $G^N(p,\kappa)$ satisfying the conditions of the statement. For each $j \in [k]$, conditional on $(W^0, W^1, \dots, W^r)$, W_j^{r+1} has distribution

$$\mathrm{Bin}\Big(p_j - (W^0 + W^1 + \ldots + W^r)_j, 1 - e^{-(W^r\kappa)_j/N}\Big). \tag{3.32}$$

The first parameter is the number of type j vertices in the graph that are not within distance r from v. For each of these vertices independently, the probability that it is connected to none of the vertices at distance r from v is $\prod_{i=1}^k (e^{-\kappa_{i,j}/N})^{W_i^r} = e^{-(W^r\kappa)_j/N}$. So,

$$\begin{aligned}\mathbb{E}\Big[W_j^{r+1} \,|\, W^0, \dots, W^r\Big] &= \Big[p_j - (W^0 + W^1 + \ldots + W^r)_j\Big] \cdot \Big(1 - e^{-(W^r\kappa)_j/N}\Big) \\ &= \left[\pi_j - \frac{(W^0 + \ldots + W^r)_j}{N}\right] \cdot N\Big(1 - e^{-(W^r\kappa)_j/N}\Big) \\ &\le \pi_j [W^r\kappa]_j = [W^r(\kappa \circ \pi)]_j.\end{aligned}$$

And so we conclude that

$$\mathbb{E}\Big[W^{r+1} \,|\, W^0, \dots, W^r\Big] \le W^r(\kappa \circ \pi). \tag{3.33}$$

Define the matrix $D^{(r)} \in \mathbb{R}^{k\times k}$ by

$$D^{(r)}_{i,j} = (\kappa \circ \pi)_{i,j} - \frac{1}{||W^r||_1}\Big[W^r(\kappa\circ\pi) - \mathbb{E}\Big[W^{r+1} \,|\, W^0, \dots, W^r\Big]\Big]_j.$$

So we may write this conditional expectation as

$$\mathbb{E}\Big[W^{r+1} \,|\, W^0, \dots, W^r\Big] = W^r D^{(r)}. \tag{3.34}$$

We define $S^r := ||W^0 + \ldots + W^r||_1$ to be the total number of vertices within radius r of the root and recall that $x - x^2/2 \le 1 - e^{-x}$ for $x \ge 0$. We can then derive a bound in the opposite direction to (3.33):

$$\mathbb{E}\Big[W_j^{r+1} \,|\, W^0, \dots, W^r\Big] \ge \left[\pi_j - \frac{S^r}{N}\right]\left([W^r\kappa]_j - \frac{([W^r\kappa]_j)^2}{2N}\right).$$

So, when $S^r \leq \chi N$, and recalling $\kappa \in [\eta, T]^{k\times k}$, we have

$$0 \leq \left(\kappa \circ \pi - D^{(r)}\right)_{i,j} \leq \frac{1}{||W^r||_1}\left[\frac{S^r}{N}[W^r\kappa]_j + \pi_j \frac{([W^r\kappa]_j)^2}{2N}\right]$$
$$0 \leq \left(\kappa \circ \pi - D^{(r)}\right)_{i,j} \leq \chi T + \frac{\chi T^2}{2} \leq \chi T^2 \leq \theta, \tag{3.35}$$

for all $i, j \in [k]$, from the assumptions (3.31) we made at the start of the proof.

Now fix some r between 0 and $N - R - 1$. We combine all of the previous ingredients to show that $\mathbb{E}\left[W^{r+R}\right]$ has direction within δ of $\mu(\kappa \circ \pi)$. We first define a version of the process W for which the matrices governing the expected one-step evolution of W always lie within $\mathbb{B}_\theta(\kappa \circ \pi)$. We do this to show that the contributions from the rare event $\{|C(v)| > \chi N\}$ are negligible as $N \to \infty$.

Let $\tilde{W}^r = W^r$. Then, inductively, for $m = 0, 1, \ldots, R - 1$, let

$$\tilde{W}^{r+m+1} = \begin{cases} W^{r+m+1} & \text{if } S^{r+m} \leq \chi N \\ \tilde{W}^{r+m}(\kappa \circ \pi) & \text{if } S^{r+m} > \chi N. \end{cases} \tag{3.36}$$

That is, $\tilde{W}$ tracks W until the first time that S exceeds χN, and thereafter evolves deterministically, with transitions given by right-multiplying by $(\kappa \circ \pi)$. Later we will be particularly interested in $\tilde{W}^{r+R}$ as r varies, so we let $Y^r := \tilde{W}^{r+R}$. (Note that for different values of r, $(\tilde{W}^{r+m})_{m\geq 0}$ are formally *different* processes.)

For $m \geq 0$, define $\mathcal{F}_{r+m} := \sigma(W^0, \ldots, W^{r+m})$. So, in particular

$$(W^0, W^1, \ldots, W^r, \tilde{W}^{r+1}, \ldots, \tilde{W}^{r+m})$$

is $\mathcal{F}_{r+m}$-measurable. Because of (3.34) and (3.36), we have

$$\mathbb{E}\left[\tilde{W}^{r+m+1} \,|\, W^0, \ldots, W^r, \tilde{W}^{r+1}, \ldots, \tilde{W}^{r+m}\right] = \tilde{W}^{r+m} D^{(r+m)}, \tag{3.37}$$

where $D^{(r+m)}$ is $\mathcal{F}_{r+m}$-measurable. On the $\mathcal{F}^{r+m}$-measurable event $\{S^{r+m} > \chi N\}$, $D^{(r+m)} = \kappa \circ \pi$, and otherwise $D^{(r+m)} \in \mathbb{B}_\theta(\kappa \circ \pi)$ from (3.35). Therefore $D^{(r+m)} \in$

$\mathbb{B}_\theta(\kappa \circ \pi)$ almost surely. We will now show that expected R-step transitions are given by a product of R matrices in a similar way, using Lemma 3.21.

Claim: For any $1 \le m \le R$, there exist $\mathcal{F}_r$-measurable matrices $D^{(1)}, \dots, D^{(m)} \in \mathbb{B}_\theta(\kappa \circ \pi)$ such that

$$\mathbb{E}\Big[\tilde{W}^{r+m} \,|\, \mathcal{F}_r\Big] = W^r D^{(1)} \dots D^{(m)}. \tag{3.38}$$

We prove the claim by induction on m. Suppose the claim is true for a particular value of m. Clearly $\mathrm{supp}\Big(\tilde{W}^{r+m}\Big)$ is finite, and for each $w \in \mathrm{supp}\Big(\tilde{W}^{r+m}\Big)$ by (3.37), we have (after a superficial change of notation - recall r is currently fixed)

$$\mathbb{E}\Big[\tilde{W}^{r+m+1} \,|\, W^0, \dots, W^r, \tilde{W}^{r+1}, \dots, \tilde{W}^{r+m-1}, \tilde{W}^{r+m} = w\Big] = w\bar{D}^{(m+1)},$$

where $\bar{D}^{(m+1)}$ is $\mathcal{F}_{r+m}$-measurable and in $\mathbb{B}_\theta(\kappa \circ \pi)$. So

$$\mathbb{E}\Big[\tilde{W}^{r+m+1} \,|\, W^0, \dots, W^r, \tilde{W}^{r+m} = w\Big] = w\mathbb{E}\Big[\bar{D}^{(m+1)} \,|\, W^0, \dots, W^r, \tilde{W}^{r+m} = w\Big].$$

The expectation on the RHS is a convex combination of elements of the convex set $\mathbb{B}_\theta(\kappa \circ \pi)$. So $D^{(m+1,w)} := \mathbb{E}\Big[\bar{D}^{(m+1)} \,|\, W^0, \dots, W^r, \tilde{W}^{r+m} = w\Big]$ is $\mathcal{F}_r$-measurable, and is almost surely in $\mathbb{B}_\theta(\kappa \circ \pi)$.

We now apply the tower law:

$$\begin{aligned}\mathbb{E}\Big[\tilde{W}^{r+m+1} \,|\, \mathcal{F}_r\Big] &= \sum_{w \in \mathrm{supp}(\tilde{W}^{r+m})} \mathbb{E}\Big[\tilde{W}^{r+m+1} \,|\, W^0, \dots, W^r, \tilde{W}^{r+m} = w\Big]\mathbb{P}\Big(\tilde{W}^{r+m} = w \,|\, \mathcal{F}_r\Big) \\ &= \sum_{w \in \mathrm{supp}(\tilde{W}^{r+m})} wD^{(m+1,w)}\mathbb{P}\Big(\tilde{W}^{r+m} = w \,|\, \mathcal{F}_r\Big).\end{aligned}$$

So by Lemma 3.21, there exists an $\mathcal{F}_r$-measurable matrix $D^{(m+1)} \in \mathbb{B}_\theta(\kappa \circ \pi)$ such that

$$\begin{aligned}\mathbb{E}\Big[\tilde{W}^{r+m+1} \,|\, \mathcal{F}_r\Big] &= \left(\sum_{w \in \mathrm{supp}(\tilde{W}^{r+m})} w\mathbb{P}\Big(\tilde{W}^{r+m} = w \,|\, \mathcal{F}_r\Big)\right)D^{(m+1)} \\ &= \mathbb{E}\Big[\tilde{W}^{r+m} \,|\, \mathcal{F}_r\Big]D^{(m+1)},\end{aligned}$$

a conditional version of (3.34). Then, using the assumed inductive hypothesis,

$$\mathbb{E}\Big[\tilde{W}^{r+m+1}\,|\,\mathcal{F}_r\Big] = W^r D^{(1)} \dots D^{(m)} D^{(m+1)}.$$

The claim (3.38) follows for all $m \leq R$ by induction. In particular, the case $m = R$ gives

$$\mathbb{E}\Big[\tilde{W}^{r+R}\,|\,\mathcal{F}_r\Big] = W^r D^{(1)} \dots D^{(R)}. \tag{3.39}$$

Since each $D^{(m)} \in \mathbb{B}_\theta(\kappa \circ \pi)$, we now have precisely the conditions to use Lemma 3.14, whenever $W^r \neq 0$. Fix $\delta > 0$, also note that $\kappa \circ \pi \in [\eta^2, T]^{k\times k}$ by assumption. The lemma specifies $R = R(\delta/2, \eta, T)$ and we conclude that

$$\Big|\Big|\mathbb{E}\Big[\tilde{W}^{r+R}\,|\,\mathcal{F}^r\Big] - \mu(\kappa\circ\pi)||\mathbb{E}[\tilde{W}^{r+R}\,|\,\mathcal{F}^r]||_1\Big|\Big|_1 \leq \frac{\delta}{2}\Big|\Big|\mathbb{E}\Big[\tilde{W}^{r+R}\,|\,\mathcal{F}^r\Big]\Big|\Big|_1,$$

almost surely (including the trivial case $W^r = 0$), and in particular,

$$\Big|\Big|\mathbb{E}\Big[\tilde{W}^{r+R}\Big] - \mu(\kappa\circ\pi)||\mathbb{E}[\tilde{W}^{r+R}]||_1\Big|\Big|_1 \leq \frac{\delta}{2}\Big|\Big|\mathbb{E}\Big[\tilde{W}^{r+R}\Big]\Big|\Big|_1. \tag{3.40}$$

Recall that r was fixed throughout, and $Y^r := \tilde{W}^{r+R}$. In particular, if $|C(v)| \leq \chi N$, then $Y^r = W^{r+R}$. Now we may sum (3.40) over r.

$$\left|\left|\mathbb{E}\left[\sum_{r=0}^{N-R-1} Y^r\right] - \mu(\kappa\circ\pi)\left|\left|\mathbb{E}\left[\sum_{m=0}^{N-R-1} Y^r\right]\right|\right|_1\right|\right|_1 \leq \frac{\delta}{2}\left|\left|\mathbb{E}\left[\sum_{r=0}^{N-R-1} Y^r\right]\right|\right|_1. \tag{3.41}$$

By considering (3.39) in the case $r = 0$, we have

$$0 < (\eta^2 - \theta)^R \leq \mathbb{E}\Big[Y^0\Big] \leq \left|\left|\mathbb{E}\left[\sum_{r=0}^{N-R-1} Y^r\right]\right|\right|_1. \tag{3.42}$$

We now deal with the case when $|C(v)| > \chi N$. By construction, we have the very crude bound,

$$||W^R + W^{R+1} + \dots + W^{N-1}||_1 \leq N.$$

Then, for each r, there are $R+1$ possibilities for the value of $Y^r = \tilde{W}^{r+R}$ in terms of W, depending on when S^{r+m} first exceeds χN, as given by (3.36). So we have another crude bound,

$$Y^r \leq W^{r+R} + W^{r+R-1}(\kappa \circ \pi) + \ldots + W^r(\kappa \circ \pi)^R,$$

since all of these quantities are non-negative. Furthermore, since all entries of $\kappa \circ \pi$ are at most T, we obtain,

$$\left|\left|\sum_{r=0}^{N-R-1} Y^r\right|\right|_1 \leq (1 + (kT) + \ldots + (kT)^R)N \leq (kT)^{R+1}N.$$

Therefore

$$\left|\left|\mathbb{E}\left[\sum_{r=0}^{N-R-1} Y^r\right] - \mathbb{E}\left[\sum_{r=R}^{N-1} W^r\right]\right|\right|_1 \leq \Big(1 + (kT)^{R+1}\Big)N\mathbb{P}(|C(v)| > \chi N). \tag{3.43}$$

By Theorem 3.9 this RHS is, for large enough N, much smaller than all the terms in (3.41) (recall from (3.42) that we have a positive lower bound on the RHS of (3.41)), and so we may replace $\mathbb{E}\Big[\sum_{r=0}^{N-R-1} Y^r\Big]$ with $\mathbb{E}\Big[\sum_{r=R}^{N-1} W^r\Big]$ to conclude

$$\left|\left|\mathbb{E}\left[\sum_{r=R}^{N-1} W^r\right] - \mu(\kappa \circ \pi)\left|\left|\mathbb{E}\left[\sum_{r=R}^{N-1} W^r\right]\right|\right|_1\right|\right|_1 \leq \delta\left|\left|\mathbb{E}\left[\sum_{r=R}^{N-1} W^r\right]\right|\right|_1, \tag{3.44}$$

for $N \geq N_0 = N_0(\delta, \eta, T) \in \mathbb{N}$, as required for (3.27), since $\sum_{r=R}^{N-1} W^r = W^{\geq R}(v)$.

Recall that $A_\chi := \{||C^{\geq R}(v)||_1 \leq \chi N\} \subseteq \{|C(v)| \leq \chi N\}$. We may add $\mathbb{1}_{A_\chi}$ to the statements of (3.41) and (3.43), from which we obtain

$$\left|\left|\mathbb{E}\left[\mathbb{1}_{A_\chi}\sum_{r=0}^{N-R-1} Y^r\right] - \mathbb{E}\left[\mathbb{1}_{A_\chi}\sum_{r=R}^{N-1} W^r\right]\right|\right|_1 \leq \Big(1 + (kT)^{R+1}\Big)N\mathbb{P}(|C(v)| > \chi N).$$

So we can apply the same argument as for (3.44) to conclude (3.28). $\square$

3.4 Proofs of technical lemmas

3.4.1 Proof of Lemma 3.12

We restate Lemma 3.12.

Lemma. For any $0 < \bar{\Lambda} < \Lambda$, and $K < \infty$ there exist $M \in \mathbb{N}$, and $\pi^{(1)}, \ldots, \pi^{(M)} \in \Pi_{\leq 1}$ and kernels $\kappa^{(1)}, \ldots, \kappa^{(M)} \in \mathbb{R}_{\geq 0}^{k \times k}$ such that

- $\rho(\kappa^{(m)} \circ \pi^{(m)}) = \bar{\Lambda}$ for each $m \in [M]$,
- for any $\pi \in \Pi_{\leq 1}$ and kernel $\kappa \in [0, K]^{k \times k}$ with $\rho(\kappa \circ \pi) \geq \Lambda$, there is some $m \in [M]$ for which $\pi^{(m)} \leq \pi$ and $\kappa^{(m)} \leq \kappa$.

Proof. The result is clear when

$$\mathbb{A}(K, \Lambda) := \Big\{(\kappa, \pi) \in [0, K]^{k \times k} \times \Pi_{\leq 1} \,:\, \rho(\kappa \circ \pi) \geq \Lambda\Big\}$$

is either empty or consists of one measure-kernel pair.

Otherwise, we view ρ as a continuous function $\mathbb{R}_{\geq 0}^{k \times k} \times \Pi_{\leq 1} \to \mathbb{R}_{\geq 0}$ via $\kappa \circ \pi$, and so $\mathbb{A}(K, \Lambda)$ is compact. Now, for any $\kappa, \kappa^0 \in \mathbb{R}_{\geq 0}^{k \times k}$, we say $\kappa \rhd \kappa^0$ if for all $i, j \in [k]$,

$$\begin{cases} \kappa_{i,j} \geq 0 & \text{when } \kappa^0_{i,j} = 0 \\ \kappa_{i,j} > \kappa^0_{i,j} & \text{when } \kappa^0_{i,j} > 0. \end{cases}$$

Then, for any $\kappa^0 \in \mathbb{R}_{\geq 0}^{k \times k}$, the set $\{\kappa \in \mathbb{R}_{\geq 0}^{k \times k} \,:\, \kappa \rhd \kappa^0\}$, is open in the subset topology induced on $\mathbb{R}_{\geq 0}^{k \times k}$. We also define the relation $\rhd$ on $\mathbb{R}^k$ in an exactly equivalent fashion.

Now, for any $(\kappa^0, \pi^0) \in [0, K]^{k \times k} \times \Pi_{\leq 1}$, with $\rho(\kappa^0 \circ \pi^0) = \bar{\Lambda}$, the set

$$\Big\{(\kappa, \pi) \in [0, K]^{k \times k} \times \Pi_{\leq 1} \,:\, \rho(\kappa \circ \pi) > \tfrac{\Lambda + \bar{\Lambda}}{2},\, \kappa \rhd \kappa^0,\, \pi \rhd \pi^0\Big\},$$

is open in $[0, K]^{k \times k} \times \Pi_{\leq 1}$, and so its restriction to $\mathbb{A}(K, \Lambda)$,

$$N(\kappa^0, \pi^0) := \Big\{(\kappa, \pi) \in \mathbb{R}_{\geq 0}^{k \times k} \times \mathbb{R}_{\geq 0}^{k} \,:\, \rho(\kappa \circ \pi) \geq \Lambda,\, \kappa \rhd \kappa^0,\, \pi \rhd \pi^0\Big\},$$

is also open in the subset topology induced on $\mathbb{A}(K,\Lambda)$. But for any $(\kappa,\pi)\in\mathbb{A}(K,\Lambda)$, with $\Lambda'=\rho(\kappa\circ\pi)$, we have

$$\rho\Big(\sqrt{\tfrac{\bar\Lambda}{\Lambda'}}\kappa\circ\sqrt{\tfrac{\bar\Lambda}{\Lambda'}}\pi\Big)=\bar\Lambda,\quad\text{and}\quad(\kappa,\pi)\in N\Big(\sqrt{\tfrac{\bar\Lambda}{\Lambda'}}\kappa,\sqrt{\tfrac{\bar\Lambda}{\Lambda'}}\pi\Big).$$

Therefore, the sets $N(\kappa^0,\pi^0)$ cover $\mathbb{A}(K,\Lambda)$. Thus there is a finite sub-cover given by some $N(\kappa^{(1)},\pi^{(1)})$, …, $N(\kappa^{(M)},\pi^{(M)})$. Certainly if $\pi\rhd\pi^{(m)}$ and $\kappa\rhd\kappa^{(m)}$, then $\pi\ge\pi^{(m)}$ and $\kappa\ge\kappa^{(m)}$, as required. □

3.4.2 Proof of Proposition 3.13

For the next two results, we denote by $S^{k\times k}([\eta,T])$, the set of $k\times k$ symmetric matrices with entries in $[\eta,T]$. For A a real symmetric positive matrix, we let $\bar\mu(A)$ be the principal left-eigenvector of A, normalised so that $||\bar\mu(A)||_2=1$. We will work with $\bar\mu(A)$ in the following result, and convert the statement to the language of $\mu(A)$ (as defined earlier) at the end.

We will also work with $\Pi_{\le 1}\cap[\eta,1]^k$, the set of sub-distributions where every component is at least η.

Lemma 3.22. Fix $0<\eta<T<\infty$. Then,

$$\lim_{R\to\infty}\sup_{A\in S^{k\times k}([\eta,T])}\sup_{v\in\Pi_{\le 1}}\left|\left|\frac{vA^R}{\rho(A)^R}-\langle v,\bar\mu(A)\rangle\bar\mu(A)\right|\right|_1=0. \tag{3.45}$$

Proof. For a real positive symmetric matrix A, we define

$$\Lambda_2(A):=\sup\{|\lambda|:\lambda\text{ an eigenvalue of }A,\ \lambda\neq\rho(A)\},$$

to be the absolute value of the 'second-largest' eigenvalue of A, which is strictly less than $\rho(A)$. But $\rho(A)$ and $\Lambda_2(A)$ are well-defined and continuous on the compact domain $S^{k\times k}([\eta,T])$. This continuity can be shown by considering the characteristic polynomial of A and applying standard results (see [72] and references therein) concerning the roots of monic polynomials under continuously varying the coefficients. Then

$$\theta(\eta, T) := \sup\left\{\frac{\Lambda_2(A)}{\rho(A)} : A \in S^{k\times k}([\eta, T])\right\} < 1. \tag{3.46}$$

Now, let $\{\bar{\mu}(A), \mu^{(2)}(A), \ldots, \mu^{(k)}(A)\}$ be a set of orthonormal eigenvectors of A, where $\bar{\mu}(A)$ corresponds to the Perron root $\rho(A)$. As usual, any $v \in \mathbb{R}^k$ can be expressed as

$$v = \langle v, \bar{\mu}(A)\rangle\bar{\mu}(A) + \langle v, \mu^{(2)}(A)\rangle\mu^{(2)}(A) + \ldots + \langle v, \mu^{(k)}(A)\rangle\mu^{(k)}(A),$$

and so

$$\left|\left|\frac{vA^R}{\rho(A)^R} - \langle v, \bar{\mu}(A)\rangle\bar{\mu}(A)\right|\right|_1 \leq \theta(\eta, T)^R \sum_{i=2}^{k} \left|\langle v, \mu^{(i)}(A)\rangle\right| ||\mu^{(i)}(A)||_1.$$

But since $v \in \Pi_{\leq 1}$,

$$\left|\langle v, \mu^{(i)}(A)\rangle\right| \leq ||\mu^{(i)}(A)||_1 \leq \sqrt{k},$$

by Cauchy–Schwarz, since $||\mu^{(i)}(A)||_2 = 1$. Therefore

$$\left|\left|\frac{vA^R}{\rho(A)^R} - \langle v, \bar{\mu}(A)\rangle\bar{\mu}(A)\right|\right|_1 \leq \theta(\eta, T)^R \cdot (k-1)\sqrt{k},$$

and the required result (3.45) follows. □

We can now restate and prove Proposition 3.13.

Proposition. Fix $0 < \eta < T < \infty$. Then,

$$\lim_{R\to\infty} \sup_{\substack{\pi \in \Pi_{\leq 1}\cap[\eta,1]^k \\ \kappa \in [\eta,T]^{k\times k}}} \sup_{v \in \Pi_1} \left|\left|\frac{v(\kappa\circ\pi)^R}{||v(\kappa\circ\pi)^R||_1} - \mu(\kappa\circ\pi)\right|\right|_1 = 0. \tag{3.5}$$

Proof. Instead of considering $\kappa \circ \pi$, we will study $\kappa \bullet \pi$, defined for $\kappa \in \mathbb{R}^{k\times k}, \pi \in \mathbb{R}^k_+$ by

$$[\kappa \bullet \pi]_{i,j} := \sqrt{\pi_i}\kappa_{i,j}\sqrt{\pi_j}. \tag{3.47}$$

The matrix $\kappa \bullet \pi$ is real and symmetric, which makes a treatment of its spectrum easier. First, we note that if v is *any* left-eigenvector of $\kappa \circ \pi$, with eigenvalue λ, then

$$\sum_{i=1}^{k} \left(\frac{v_i}{\sqrt{\pi_i}}\right)[\kappa \bullet \pi]_{i,j} = \sum_{i=1}^{k} v_i \kappa_{i,j} \sqrt{\pi_j} = \lambda \frac{v_j}{\sqrt{\pi_j}}.$$

That is $(v_i/\sqrt{\pi_i})$ is an eigenvector of $\kappa \bullet \pi$, also with eigenvalue λ. Therefore the spectrum of $\kappa \circ \pi$ is the same as the spectrum of $\kappa \bullet \pi$. In particular, the Perron roots of $\kappa \circ \pi$ and $\kappa \bullet \pi$ are the same, and $\mu(\kappa \bullet \pi)_i = C\mu(\kappa \circ \pi)_i/\sqrt{\pi_i}$, where C is a positive constant chosen to ensure consistent normalisation.

But then

$$[v(\kappa \circ \pi)^R]_j = \left[\left(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}\right)(\kappa \bullet \pi)^R\right]_j \sqrt{\pi_j}. \tag{3.48}$$

Note that if $v \in \Pi_1$, then $(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}) \in \Pi_{\le \eta^{-1/2}}$, and certainly $\kappa \bullet \pi \in S^{k\times k}([\eta^2, T])$. The statement (3.45) still holds after replacing the supremum over $v \in \Pi_{\le 1}$ with a supremum over $v \in \Pi_{\le \eta^{-1/2}}$. So we can treat the RHS of (3.48), since

$$\left\| \left(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}\right)\frac{(\kappa \bullet \pi)^R}{\rho^R} - \left\langle \left(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}\right), \bar{\mu}(\kappa \bullet \pi)\right\rangle \bar{\mu}(\kappa \bullet \pi)\right\|_1 \to 0,$$

as $R \to \infty$, uniformly across the set of (π, κ) under consideration, and $v \in \Pi_1$. Note that for each $j \in [k]$, we have $\sqrt{\pi_j} \in [\sqrt{\eta}, 1]$. Therefore, uniformly in the same sense,

$$\frac{v(\kappa \circ \pi)^R}{\rho^R} \quad \to \quad \left\langle \left(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}\right), \bar{\mu}(\kappa \bullet \pi)\right\rangle \left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \dots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k}\right),$$

as $R \to \infty$, and so also

$$\left\|\frac{v(\kappa \circ \pi)^R}{\rho^R}\right\|_1 \quad \to \quad \left\| \left\langle \left(\frac{v_1}{\sqrt{\pi_1}}, \dots, \frac{v_k}{\sqrt{\pi_k}}\right), \bar{\mu}(\kappa \bullet \pi)\right\rangle \left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \dots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k}\right)\right\|_1.$$

We want to show that this limiting quantity has a positive lower bound, so that we can take a limit of the quotients $\frac{v(\kappa\circ\pi)^R}{\|[v(\kappa\circ\pi)^R]\|_1}$. We first note that from (3.4), $\rho(\kappa \bullet \pi) \le kT$, since $\kappa \bullet \pi \in S^{k\times k}([\eta^2, T])$. Then, similarly to (3.20), we can bound the components of

$\mu(\kappa \bullet \pi)$ below since

$$\mu_j = \frac{1}{\rho(\kappa \bullet \pi)} \sum_{i \in [k]} \mu_i [\kappa \bullet \pi]_{i,j} \geq \frac{1}{kT} \sum_{i \in [k]} \mu_i \eta^2 = \frac{\eta^2}{kT}.$$

Note also that $\bar{\mu}(\kappa \bullet \pi) \geq \mu(\kappa \bullet \pi)$. So, since $v \in \Pi_1$ and $\sqrt{\pi_i} \leq 1$, we obtain

$$\left\langle \left(\frac{v_1}{\sqrt{\pi_1}}, \ldots, \frac{v_k}{\sqrt{\pi_k}} \right), \bar{\mu}(\kappa \bullet \pi) \right\rangle \geq \frac{\eta^2}{kT},$$

and, since $\sqrt{\pi_i} \geq \sqrt{\eta}$, we also obtain

$$\left|\left| \left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k} \right) \right|\right|_1 \geq \sqrt{\eta}.$$

Thus

$$\left|\left| \left\langle \left(\frac{v_1}{\sqrt{\pi_1}}, \ldots, \frac{v_k}{\sqrt{\pi_k}} \right), \bar{\mu}(\kappa \bullet \pi) \right\rangle \left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k} \right) \right|\right|_1 \geq \frac{\eta^{5/2}}{kT} > 0.$$

So we obtain

$$\frac{v(\kappa \circ \pi)^R}{||v(\kappa \circ \pi)^R||_1} \quad \to \quad \frac{\left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k} \right)}{\left|\left| \left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k} \right) \right|\right|_1}.$$

But

$$\left(\bar{\mu}_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \bar{\mu}_k(\kappa \bullet \pi)\sqrt{\pi_k} \right) \propto \left(\mu_1(\kappa \bullet \pi)\sqrt{\pi_1}, \ldots, \mu_k(\kappa \bullet \pi)\sqrt{\pi_k} \right) \propto \mu(\kappa \circ \pi),$$

so we have shown

$$\frac{v(\kappa \circ \pi)^R}{||[v(\kappa \circ \pi)^R]||_1} \quad \to \quad \mu(\kappa \circ \pi),$$

as $R \to \infty$, uniformly across $v \in \Pi_1$, and $\kappa \in [\eta, T]^{k \times k}$ and $\pi \in \Pi_1$ such that $\pi_i \geq \eta$, exactly as required. $\square$

3.4.3 Proof of Lemma 3.14

Recall the definition (3.6)

$$\mathbb{B}_\theta(A) := \{B \in \mathbb{R}_+^{k\times k} \,:\, |B_{i,j} - A_{i,j}| \le \theta,\, \forall i,j \in [k]\},$$

of the set of positive kernels whose entries differ from those of A by at most θ. We now restate Lemma 3.14.

Lemma. For all $0 < \eta < T < \infty$ with $\eta < 1$, and $\delta > 0$, there exists $\theta = \theta(\delta, \eta, T) \in (0, \eta^2)$ and $R = R(\delta, \eta, T) < \infty$ such that

$$\left|\left| \frac{vD^{(1)} \dots D^{(R)}}{||vD^{(1)} \dots D^{(R)}||_1} - \mu(\kappa \circ \pi) \right|\right|_1 < \delta, \tag{3.7}$$

for all $v \in \mathbb{R}^k_{\ge 0} \backslash \{0\}$, $\kappa \in [\eta, T]^{k\times k}$, $\pi \in \Pi_{\le 1} \cap [\eta, 1]^k$, and $D^{(1)}, \dots, D^{(R)} \in \mathbb{B}_\theta(\kappa \circ \pi)$.

Proof. For now we fix $\theta \in (0, \eta^2)$, and will take this small enough at the end. Then, for any $A \in [\eta^2, T]^{k\times k}$ and $D^{(1)}, \dots, D^{(R)} \in \mathbb{B}_\theta(A)$,

$$(D^{(1)} D^{(2)} \dots D^{(R)})_{i,j} = \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R D^{(r)}_{i_{r-1},i_r} \le \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R (A_{i_{r-1},i_r} + \theta).$$

Therefore, defining $\bar{D} := D^{(1)} \cdots D^{(R)}$, since $\theta < \eta^2 < \eta < T$,

$$(\bar{D} - A^R)_{i,j} \le k^{R-1}(2^R - 1) \cdot \theta T^{R-1}.$$

Similarly, for a lower bound

$$(A^R - \bar{D})_{i,j} \le \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R A_{i_{r-1},i_r} - \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R (A_{i_{r-1},i_r} - \theta).$$

The RHS is a polynomial in θ whose coefficients have alternating signs, and so we can bound using the associated polynomial with every coefficient positive:

$$(A^R - \bar{D})_{i,j} \le \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R (A_{i_{r-1},i_r} + \theta) - \sum_{i=i_0,i_1,\dots,i_R=j} \prod_{r=1}^R A_{i_{r-1},i_r}.$$

That is,

$$\left|(\bar{D} - A^R)_{i,j}\right| \leq k^{R-1}(2^R - 1) \cdot \theta T^{R-1}.$$

Since the fraction in (3.7) is unchanged under positive scalar multiplication of v, it suffices to show the result for $v \in \Pi_1$. For any $v \in \Pi_1$:

$$||v\bar{D} - vA^R||_1 \leq k^R(2^R - 1) \cdot \theta T^{R-1}.$$

For (3.7) we need to control the distance between the normalised vectors instead. Observe first that for each i, $||vD^{(i)}||_1 \in [k(\eta - \theta), k(T + \theta)]$, whenever $v \in \Pi_1$. Thus $||v\bar{D}||_1, ||vA^R||_1 \in [(k(\eta - \theta))^R, (k(T + \theta))^R]$. From the triangle inequality,

$$\begin{aligned}\left|\left|\frac{v\bar{D}}{||v\bar{D}||_1} - \frac{vA^R}{||vA^R||_1}\right|\right|_1 &\leq \left|\left|\frac{v\bar{D} - vA^R}{||v\bar{D}||_1}\right|\right|_1 + \left|\left|\frac{vA^R}{||v\bar{D}||_1} - \frac{vA^R}{||vA^R||_1}\right|\right|_1 \\ &\leq \frac{||v\bar{D} - vA^R||_1}{||v\bar{D}||_1} + ||vA^R||_1\left|\frac{1}{||v\bar{D}||_1} - \frac{1}{||vA^R||_1}\right| \\ &\leq \frac{||v\bar{D} - vA^R||_1}{||v\bar{D}||_1} + ||vA^R||_1\frac{\left|||v\bar{D}||_1 - ||vA^R||_1\right|}{||v\bar{D}||_1||vA^R||_1} \\ &\leq \frac{2||v\bar{D} - vA^R||_1}{||v\bar{D}||_1},\end{aligned} \tag{3.49}$$

so for $v \in \Pi_1$,

$$\left|\left|\frac{v\bar{D}}{||v\bar{D}||_1} - \frac{vA^R}{||vA^R||_1}\right|\right|_1 \leq \frac{2(2^R - 1) \cdot \theta T^{R-1}}{(\eta - \theta)^R}. \tag{3.50}$$

Finally, we take $A = \kappa \circ \pi$. Proposition 3.13 determines a value of R such that for all $\kappa \in [\eta, T]^{k \times k}$, $\pi \in \Pi_{\leq 1} \cap [\eta, 1]^k$, and $v \in \Pi_1$, taking $A = \kappa \circ \pi$, we have

$$\left|\left|\frac{vA^R}{||vA^R||_1} - \mu(A)\right|\right|_1 \leq \frac{\delta}{2}.$$

Combining with (3.50) and taking θ small enough, the result follows after extending from $v \in \Pi_1$ to $v \in \mathbb{R}^k_{\geq 0}\backslash\{0\}$. □

3.4.4 Proof of Lemma 3.15

We now restate Lemma 3.15.

Lemma. Let $\mathbb{A}$ be a compact subset of $\mathbb{R}_{\geq 0}^{k\times k}$ with the property that for any $A \in \mathbb{A}$, the Perron root of A is simple. Then there exists a constant $C(\mathbb{A}) < \infty$ such that, for all matrices $A, A' \in \mathbb{A}$,

$$||\mu(A) - \mu(A')||_1 \leq C(\mathbb{A}) \max_{i,j\in[k]} |A_{i,j} - A'_{i,j}|. \tag{3.8}$$

In particular, for any $0 < \eta < T < \infty$, there exists $C(\eta, T) < \infty$ such that, for all matrices $A, A' \in [\eta, T]^{k\times k}$,

$$||\mu(A) - \mu(A')||_1 \leq C(\eta, T) \max_{i,j\in[k]} |A_{i,j} - A'_{i,j}|. \tag{3.9}$$

Proof. Note that (3.9) follows by taking $\mathbb{A} = [\eta, T]^{k\times k}$ in (3.8). To show (3.8), we use the following result about the local smoothness of eigenvalues and eigenvectors as the matrix varies in the neighbourhood of a matrix with a simple eigenvalue.

THEOREM ([48], §3.9, Theorem 8)**.** Let ρ_0 be a simple eigenvalue of a matrix $A_0 \in \mathbb{C}^{k\times k}$, and μ_0 an associated left-eigenvector satisfying $\mu_0^\dagger \mu_0 = 1$. Then, there exists a neighbourhood of $N(A_0) \subseteq \mathbb{C}^{k\times k}$ of A_0, and functions $\rho : N(A_0) \to \mathbb{C}$ and $\bar{\mu} : N(A_0) \to \mathbb{C}^k$, such that

- $\rho(A_0) = \rho_0$ and $\bar{\mu}(A_0) = \mu_0$,
- $\bar{\mu}(A)A = \rho(A)\bar{\mu}(A)$, and $\mu_0^\dagger \bar{\mu}(A) = 1$ for all $A \in N(A_0)$,
- ρ and $\bar{\mu}$ are infinitely differentiable on $N(A_0)$.

If we take $A_0 \in \mathbb{A}$, and $\mu_0 = \mu(A)$, then it follows that $\bar{\mu}$ is locally Lipschitz as a function $N(A_0) \cap \mathbb{A} \to \mathbb{R}_+^k$. In this statement $\bar{\mu}(A)$ differs from our definition $\mu(A)$ by a normalising factor, that varies in $N(A_0)$. However, the choice $\bar{\mu}$ satisfies $\bar{\mu}(A_0)^T\bar{\mu}(A_0) = 1$, and so for each $i \in [k]$, $\bar{\mu}_i(A_0) \leq 1$. Therefore, for any $A \in N(A_0) \cap \mathbb{A}$,

$$||\bar{\mu}(A)||_1 \geq \bar{\mu}(A_0)^T\bar{\mu}(A) = 1. \tag{3.51}$$

Now, for $A, A' \in N(A_0) \cap \mathbb{A}$,

$$||\mu(A) - \mu(A')||_1 = \left|\left|\frac{\bar{\mu}(A)}{||\bar{\mu}(A)||_1} - \frac{\bar{\mu}(A')}{||\bar{\mu}(A')||_1}\right|\right|.$$

Therefore, as in (3.49),

$$||\mu(A) - \mu(A')||_1 \leq \frac{2||\bar{\mu}(A) - \bar{\mu}(A')||_1}{||\bar{\mu}(A)||_1} \overset{(3.51)}{\leq} 2||\bar{\mu}(A) - \bar{\mu}(A')||_1.$$

Since $\bar{\mu}$ is locally Lipschitz on $N(A_0) \cap \mathbb{A}$, it follows that μ is also locally Lipschitz on $N(A_0) \cap \mathbb{A}$. Thus μ is Lipschitz on $\mathbb{A}$ by compactness. □

3.4.5 Proof of Lemma 3.19

We now restate Lemma 3.19.

Lemma. Fix $\eta > 0$, and set $\theta(\eta) := \frac{\eta^4}{8}$. Then there exists a function $c : (0, \theta(\eta)) \to (0, \eta/2]$ such that:

- for all $\epsilon \in (0, \theta(\eta))$, whenever we take $\pi \in \Pi_{\leq 1}$ satisfying $\pi_i \geq \eta$ for all $i \in [k]$, and $\kappa \in [\eta, \infty)^{k \times k}$ such that $\rho(\kappa \circ \pi) \leq 1 + \epsilon$, then for any $\pi' \in \Pi_{\leq 1}$ satisfying

$$\pi' \leq \pi, \quad \text{and} \quad ||\pi - \pi'||_1 \geq c(\epsilon),$$

  we have $\rho(\kappa \circ \pi') \leq 1 - \epsilon$;

- as $\epsilon \to 0$, $c(\epsilon) \to 0$.

Proof. For each $\epsilon > 0$, consider any choice of $c(\epsilon) \in (0, \eta/2]$ for now, and take κ, π, π' satisfying the conditions. By considering for example a suitable linear combination of π and π', there exists $\pi'' \in \Pi_{\leq 1}$ such that $\pi' \leq \pi'' \leq \pi$ and $||\pi - \pi''|| = c(\epsilon)$. Note that since $c(\epsilon) \leq \eta/2$, $\pi''_i \geq \eta/2$ for every i. To avoid confusion when we multiply on the right by the matrices themselves, we let $v = \mu(\kappa \circ \pi)$ and $v'' = \mu(\kappa \circ \pi'')$. Then,

$$v''_j = \sum v''_i \kappa_{i,j} \pi''_j \geq \frac{\eta^2}{2}, \quad \forall j \in [k].$$

We now consider the Collatz–Wielandt formula (3.3) applied to the matrix $\kappa \circ \pi$ and v'' (which is not generally an eigenvector of $\kappa \circ \pi$). We conclude that there exists $i \in [k]$ such that $[v''(\kappa \circ \pi)]_i \leq (1+\epsilon)v''_i$. Thus

$$\begin{aligned}
\rho(\kappa \circ \pi'')v''_i = [v''(\kappa \circ \pi'')]_i &= [v''(\kappa \circ \pi)]_i - [v''(\kappa \circ (\pi - \pi''))]_i \\
&\leq (1+\epsilon)v''_i - \frac{\eta^2}{2} \cdot \eta \cdot c(\epsilon) \\
&\leq (1+\epsilon)v''_i - \frac{\eta^2}{2} \cdot \eta \cdot c(\epsilon)v''_i.
\end{aligned}$$

Note that since $\epsilon \leq \theta(\eta) = \frac{\eta^4}{8}$, we have $\frac{4\epsilon}{\eta^3} \leq \frac{\eta}{2}$. So we may choose $c(\epsilon) \in [\frac{4\epsilon}{\eta^3}, \frac{\eta}{2}]$, and in particular $c(\epsilon) = \frac{4\epsilon}{\eta^3}$ is suitable. Corollary 3.11 then shows

$$\rho(\kappa \circ \pi') \leq \rho(\kappa \circ \pi'') \leq 1 - \epsilon,$$

as required. □

Chapter 4

Frozen percolation - convergence to Smoluchowski's equations

In Section 1.1.4, we introduced multiplicative coalescence, and the corresponding Smoluchowski equations (1.4). In Section 1.3.1, we introduced Ráth's model of mean-field frozen percolation.

In this chapter, we show convergence of (v^N), the proportions of vertices in components of each size in a sequence of mean-field frozen percolation processes, to a solution of the corresponding Smoluchowski equations. In particular, we are able to show this in greater generality than Ráth's Theorem 1.2 [60], by adapting a method used by Merle and Normand [51] for a related process where components are, essentially, frozen once they reach a certain threshold size. We will use this in Chapter 5 to study a model of mean-field frozen percolation where the initial graph is an IRG with k types, as defined in Section 3.1.2.

4.1 Preliminaries

4.1.1 Definitions and results

Multiplicative Smoluchowski equations: existence and uniqueness

Recall from (1.4) the Smoluchowski equations with multiplicative kernel

$$\frac{\mathrm{d}}{\mathrm{d}t}v_k(t) = \frac{k}{2}\sum_{\ell=1}^{k-1} v_\ell(t)v_{k-\ell}(t) - kv_k(t)\sum_{\ell=1}^{\infty} v_\ell(t), \quad k \geq 1. \tag{4.1}$$

For our purposes, throughout we will always assume $v_k(t) \geq 0$. Sometimes it will be helpful to use the following notation for the total mass

$$\Phi(t) := \sum_{k=1}^{\infty} v_k(t). \tag{4.2}$$

We will use Normand and Zambotti's results [53] about the global existence and uniqueness of solutions to (4.1). The following slightly weaker summary will be sufficient for our requirements.

THEOREM 4.1 (Theorem 2.2, [53])**.** Whenever $\sum_{k\geq 1} v_k(0) < \infty$, there exists a unique solution to Smoluchowski's equations (4.1) starting from $v(0)$. For this solution, $\Phi(t)$ is uniformly continuous on $[0,\infty)$. Indeed, $\Phi(t)$ is constant on $[0, T_g]$ and strictly decreasing on $[T_g, \infty)$, where

$$T_g = \frac{1}{\sum\limits_{k\geq 1} kv_k(0)}.$$

Note. Theorem 2.2 of [53] further shows that Φ is analytic on $[0,\infty)\backslash\{T_g\}$. Some steps of the arguments in Section 4.2.2 would be easier with the stronger assumption that Φ is Lipschitz on $[0,\infty)$. We believe that Φ should be Lipschitz for those initial conditions considered in Chapter 5, but we avoid this issue by using the known, weaker condition of uniform continuity.

Mean-field frozen percolation

We recall the definition of the mean-field frozen percolation model given in Section 1.3.1. In particular, for a realisation of the model on N vertices, we defined $v_k^N(t) \geq 0$ to be the proportion of vertices which are alive and in a component of size k at time t. To allow slightly more generality, we will relax the condition that the initial graph has N vertices, but we will continue to demand for a mean-field frozen percolation process *with index* N, that each potential edge is added at rate $1/N$, and the lightning rate per vertex is $\lambda(N)$, satisfying the critical scaling

$$1/N \ll \lambda(N) \ll 1.$$

Each process (v^N) is a random non-negative element of $\mathbb{D}([0,\infty),\ell_1)$, the Skorohod space of càdlàg sequence-valued functions with bounded absolute sum. Furthermore $v_k^N(t)$ is always non-negative. Since all transitions of (v^N) are governed by independent exponential clocks, each (v^N) is a pure-jump Markov process, with natural filtration $(\mathcal{F}^N(t), t \geq 0)$. From now onwards in this chapter, we will make no reference to any underlying graph structure, and view frozen percolation as *multiplicative coalescence with linear deletion*.

We assume hereafter that a sequence of mean-field frozen percolation processes (v^N) is given, where each v^N has index N, and the initial state $v^N(0)$ has non-negative entries and may be random. As before, we set $\Phi^N(t) := \sum_{k\geq 1} v_k^N(t)$. In this chapter, we will prove the following result about convergence of the processes (v^N). Recall from Section 1.3.1 that this is a mild generalisation of Ráth's Theorem 1.2 [60], where the initial conditions $v^N(0)$ are deterministic and have finite support. (We remark that it is the second of these that requires substantial work to generalise.) In what follows, we will always assume $v(0)$ has non-negative entries.

THEOREM 4.2. We assume $v^N(0) \stackrel{d}{\to} v(0) \in \ell_1$, and let v be the unique solution to (4.1) started from $v(0)$, as given by Theorem 4.1. Then $v^N \to v$ in distribution in $\mathbb{D}([0,\infty),\ell_1)$, with respect to the uniform topology. In particular, $\Phi^N \to \Phi$ in distribution in $\mathbb{D}([0,\infty))$ with respect to the uniform topology.

For much of the chapter, we will work with c and (c^N), defined by

$$c_k(t) := \frac{1}{k}v_k(t), \quad c_k^N(t) := \frac{1}{k}v_k^N(t), \quad k \geq 1,\, t \in [0,\infty), \tag{4.3}$$

which represent the density of blocks with size k, rather than the proportion of vertices on blocks with size k. Clearly $v(0) \in \ell_1$ implies $c(0) \in \ell_1$, and furthermore $v^N(0) \stackrel{d}{\to} v(0)$ in ℓ_1 implies $c^N(0) \stackrel{d}{\to} c(0)$ in ℓ_1.

In the remainder of this section, we introduce the notation and show a tightness result for the processes (c^N). We prove convergence of (c^N) in Section 4.2.1, at times following closely the approach of Merle and Normand [51]. In Section 4.2.2 we establish convergence of (Φ^N) and, from this, convergence of (v^N).

4.1.2 Martingale formulation

Each process c^N corresponding to a mean-field frozen percolation process with index N is Markov. We will prove convergence of sequences of such processes (c^N) via a martingale formulation in the manner of Stroock and Varadhan [67].

With direct reference to [51], we define the product

$$\langle f, g\rangle = \sum_{k\geq 1} f(k)g(k), \quad \forall f,g : \mathbb{N} \to \mathbb{R}_{\geq 0}.$$

In a mild abuse of notation, we view both k and $c_k(t)$ as functions of k of this form. So, for example, $\langle c^N(t), k\rangle = \sum_{k\geq 1} v_k^N(t) = \Phi^N(t)$ is the proportion of vertices alive at time t; and $\langle c^N(t), 1\rangle = \sum_{k\geq 1} c_k^N(t)$ is the rescaled number of alive components. The following lemma about these processes is clear from the dynamics.

Lemma 4.3. For all N, both $\langle c^N(\cdot), 1\rangle$ and $\langle c^N(\cdot), k\rangle$ are almost surely non-increasing.

The definition which follows is analogous to the definition of a *weak* solution to an SDE. We say $(c_k(t), k \geq 1)$, a collection of non-negative continuous functions, is a *solution to Smoluchowski's equation* with initial condition $c(0) \in [0,\infty)^{\mathbb{N}}$ if

- for every $t \geq 0$, $\int_0^t \langle c(t), k\rangle^2 \mathrm{d}s < \infty$;
- for every $t \geq 0$ and bounded $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$

$$\langle c(t), f\rangle - \langle c(0), f\rangle = \frac{1}{2}\int_0^t \sum_{k,\ell\geq 1} k\ell c_k(s)c_\ell(s)[f(k+\ell) - f(k) - f(\ell)]\mathrm{d}s. \quad (4.4)$$

Note that (4.4) is classically defined for f with compact support, but Merle and Normand show in Lemma 2.2 [51] that this immediately extends to bounded f. We can recover (1.3) by taking $f = \mathbb{1}_{\{k\}}$ and differentiating (4.4). Conversely, (4.4) for bounded f follows from (1.3) by linearity.

4.1.3 The generator of c^N

During a frozen percolation process with index N, there are two types of event which might occur.

- Two components of sizes k and ℓ might coalesce. In the graph setting, this corresponds to the arrival of an edge between two hitherto disjoint components. This has the effect of creating a component of size $k+\ell$, while removing a component of each size k,ℓ, so let

  $$\Delta^N(k,\ell) := \frac{1}{N}\Big[\mathbb{1}_{\{k+\ell\}} - \mathbb{1}_{\{k\}} - \mathbb{1}_{\{\ell\}}\Big].$$

  Now define:

  $$\theta^N_{k,\ell}(\eta) := \begin{cases} k\ell\eta(k)\eta(\ell)N & \text{if } k \neq \ell \\ k^2\eta(k)\Big[\eta(k) - \frac{1}{N}\Big]N & \text{if } k = \ell. \end{cases}$$

  Then if (c^N) is in state $\eta \in \ell_1$, it jumps to state $\eta + \Delta^N(k,\ell)$ at rate $\theta^N_{k,\ell}(\eta)$ when $k \neq \ell$ and to state $\eta + \Delta^N(k,k)$ at rate $\frac{1}{2}\theta^N_{k,k}(\eta)$.

- A component may be frozen. Using bars to denote transitions arising from freezing, we let $\bar{\Delta}^N(k) := -\frac{1}{N}\mathbb{1}_{\{k\}}$. Then, if (c^N) is in state $\eta \in \ell_1$, it jumps to state $\eta + \bar{\Delta}^N(k)$ at rate $\bar{\theta}_k^N(\eta) := k\eta(k)N\lambda(N)$, where $\lambda(N)$ is the freezing rate per vertex.

Hence (c^N) has generator

$$\begin{aligned} G^N F(\eta) = \frac{1}{2}\sum_{k,\ell=1}^{N} \Big[F\Big(\eta + \Delta^N(k,\ell)\Big) - F(\eta)\Big]\theta^N_{k,\ell}(\eta) \\ + \sum_{k=1}^{N} \Big[F\Big(\eta + \bar{\Delta}^N(k)\Big) - F(\eta)\Big]\bar{\theta}^N_k(\eta), \end{aligned} \tag{4.5}$$

for any $F : \ell_1 \to \mathbb{R}$. From the definition of the generator of a Markov process (see, for example, Theorem 1.6 in Section 7 of [19]), we have that

$$M^{N,F}(t) := F\Big(c^N(t)\Big) - F\Big(c^N(0)\Big) - \int_0^t G^N F\Big(c^N(s)\Big)\mathrm{d}s \tag{4.6}$$

is a martingale for any bounded $F : \ell_1 \to \mathbb{R}$. It will be sufficient for our purposes to consider linear test functions, that is $F(\eta) = \langle f, \eta\rangle$ for some $f : \mathbb{N} \to \mathbb{R}$. We can then rewrite (4.5) as

$$\begin{aligned} G^N F(\eta) = \frac{1}{2}\sum_{k,\ell=1}^{N} [f(k+\ell) - f(k) - f(\ell)]k\ell\eta(k)\eta(\ell) \\ - \frac{1}{N}\sum_{k=1}^{N}[f(2k) - 2f(k)]k^2\eta(k) - \lambda(N)\sum_{k=1}^{N} f(k)k\eta(k). \end{aligned} \tag{4.7}$$

When F is linear, it is either identically zero, or unbounded. However, $F(c^N(\cdot))$ *is* bounded in the following two cases:

- when $F(\eta) = \langle f, \eta\rangle$ for some bounded $f : \mathbb{N} \to \mathbb{R}$, since

$$\Big|F\Big(c^N(t)\Big)\Big| \leq \sup_{k\in\mathbb{N}} |f(k)|\langle c^N(t), 1\rangle \overset{\text{Lemma 4.3}}{\leq} \sup_{k\in\mathbb{N}} |f(k)|\langle c^N(0), 1\rangle < \infty.$$

- When $F(\eta) = \langle \eta, k\rangle$, then $F(c^N(t)) = \langle c^N(t), k\rangle$, which is bounded by Lemma 4.3 directly.

Thus for both of these cases, $M^{N,F}(t)$ as in (4.6) is also a martingale.

4.1.4 Tightness

The following lemma is directly equivalent to Lemma 2.5 in [51], with the extra condition that we may allow $c^N(0)$ to be random without affecting the proof.

Lemma 4.4. Assume that $c(0) \in \ell_1$ satisfies $\langle c(0), k\rangle < \infty$. Take $M < \infty$ and assume that the initial distributions $(c^N(0))$ of a family of frozen percolation processes satisfy $\langle c^N(0), k\rangle \leq M$ a.s. and $c^N(0) \xrightarrow{d} c(0)$ as $N \to \infty$. Then the family of processes (c^N) is tight in $\mathbb{D}([0,\infty), \ell_1)$, and any limit point is a continuous function from $[0,\infty)$ to ℓ_1.

Proof. Since ℓ_1 is complete and separable, it will suffice to check Aldous's tightness conditions, with reference to Theorem §3.7.2 of [24]. It is clear that the uniform boundedness (in probability) condition

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\in\mathbb{R}_{\geq 0}} ||c^N(t)||_1 > K\right) \to 0, \tag{4.8}$$

holds since $||c^N(t)||_1$ is non-increasing in t, and $||c^N(0)||_1$ is uniformly bounded in probability.

The second condition demands that a large jump is asymptotically unlikely to occur in any given small time interval. Formally, we require that for all $\epsilon, T > 0$, there exists $\delta > 0$ such that for any τ^N, an $\mathcal{F}^N$-stopping time with $\tau^N \leq T$, we have

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{\eta\in[0,\delta]} |c^N(\tau^N+\eta) - c^N(\tau^N)| \geq \epsilon\right) \leq \epsilon. \tag{4.9}$$

Jumps of c^N caused by freezing have size $1/N$ in ℓ_1, and jumps caused by coalescence have size at most $3/N$. Therefore,

$$\begin{aligned}\mathbb{E}\left[\sup_{\eta\in[0,\delta]} |c^N(\tau^N+\eta) - c^N(\tau^N)| \,\Big|\, \mathcal{F}^N(\tau^N)\right] &\leq \frac{\lambda(N)\delta}{N}\langle Nc^N(\tau^N), k\rangle \\ &\quad + \frac{3}{N}\cdot\frac{\delta}{N}\langle Nc^N(\tau^N), k\rangle^2 \qquad (4.10)\\ &\leq \lambda(N)\delta\langle c^N(0), k\rangle + 3\delta\langle c^N(0), k\rangle^2 \quad \text{a.s.}\end{aligned}$$

$$\leq \lambda(N)\delta M + 3\delta M^2 \quad \text{a.s.}$$

The condition (4.9) follows from Markov's inequality, so together with (4.8), we have tightness. Continuity of the limit points follows, since, in addition to (4.9), the jumps have size (in ℓ_1) at most $3/N$, which vanishes as $N \to \infty$. □

Remark. This lemma illustrates the advantage of working with $(c^N(\cdot))$ rather than $(v^N(\cdot))$, as the sizes of the jumps of the latter family of processes do not obviously vanish as $N \to \infty$. (Though this will be a consequence of Theorem 4.2.)

4.2 Convergence

We prove Theorem 4.2. We assume that $v(0)$ and a sequence (v^N) are given, satisfying the conditions in the statement of Theorem 4.2. We will work with the associated $c(0)$ and sequence (c^N), as defined by (4.3).

4.2.1 Convergence of (c^N)

We will show that any limit point of (c^N) in $\mathbb{D}([0,\infty),\ell_1)$ solves Smoluchowski's equation. We will do this by showing that for any bounded F, the evolution of $F(c^N(\cdot))$ is dominated, for large N, by the small coalescence terms in (4.7). This is achieved by bounding in expectation the amount of mass in large components on compact time-intervals.

We consider a continuous limit process c and invoke Skorohod's representation theorem on the relevant subsequence, which for ease of notation we assume to be $\mathbb{N}$. So we may assume that we work in a probability space $(\mathbb{P},\mathcal{F},\Omega)$ such that

$$c^N \to c \quad \mathbb{P}\text{-a.s. in } \mathbb{D}([0,\infty),\ell_1), \tag{4.11}$$

with respect to the uniform topology, and where c is almost surely continuous. Throughout, we assume $T > 0$ is fixed, and consider the time-interval $[0,T]$. We will show that c satisfies (4.4) on $[0,T]$.

We consider the norm on $L^1(\mathbb{P}\otimes \mathbb{1}_{[0,T]}\mathrm{d}t)$ given by

$$||g|| = \mathbb{E}\left[\int_0^T |g(t)|\mathrm{d}t\right].$$

Note immediately, that if $g \in L^1(\mathbb{P}\otimes \mathbb{1}_{[0,T]}\mathrm{d}t)$, and we define $h(t) = \int_0^t g(s)\mathrm{d}s$ for $t \in [0,T]$, then $h \in L^1(\mathbb{P}\otimes \mathbb{1}_{[0,T]}\mathrm{d}t)$ and

$$\begin{aligned} ||h|| &= \mathbb{E}\left[\int_0^T \left|\int_0^t g(s)\mathrm{d}s\right|\mathrm{d}t\right] \\ &\leq \mathbb{E}\left[\int_0^T \int_0^t |g(s)|\mathrm{d}s\mathrm{d}t\right]. \\ &\leq T\mathbb{E}\left[\int_0^T |g(s)|\mathrm{d}s\right] = T||g||. \end{aligned} \tag{4.12}$$

We show first that the proportion of time during which there is a block of size $\Theta(N)$ is small for large N.

Lemma 4.5. Whenever the family of processes (c^N) satisfies

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\langle c^N(0), k\rangle\Big] < \infty,$$

we have

$$\lim_{N\to\infty} \frac{1}{N}||\langle c^N(\cdot), k^2\rangle|| = 0. \tag{4.13}$$

Proof. Let $F : \ell_1 \to \mathbb{R}$ be defined by $F(\eta) = \langle \eta, k\rangle$. Then, as in Section 4.1.3, for each N, the process $M^{N,F}(t)$ defined by (4.6) is a martingale. But for this choice of F, corresponding to $f(k) = k$, the first two terms in (4.7) vanish, and so we obtain that

$$\mathbb{E}\Big[\langle c^N(0), k\rangle\Big] - \mathbb{E}\Big[\langle c^N(t), k\rangle\Big] = \lambda(N)\mathbb{E}\Big[\int_0^t \langle c^N(s), k^2\rangle \mathrm{d}s\Big]. \tag{4.14}$$

Therefore, by Lemma 4.3,

$$0 \leq ||\langle c^N(\cdot), k\rangle|| \leq \frac{\mathbb{E}\Big[\langle c^N(0), k^2\rangle\Big]}{\lambda(N)}.$$

The required result (4.13) follows from the critical scaling (4.3) of $\lambda(N)$. □

We now show that the rescaled number of blocks with size at least b is small in norm $||\cdot||$, for large b and large N. We use the same test function F^b as Merle and Normand in Lemma 2.5 of [51], but our Lemma 4.5 simplifies the rest of the proof, without the requirement to truncate the component sizes.

Lemma 4.6. We assume that the family of processes (c^N) satisfies $\mathbb{E}\Big[\langle c^N(0), k\rangle\Big] \leq M$ for some $M < \infty$. Then, for every integer $b \geq 1$,

$$\limsup_{N\to\infty} \left|\left|\sum_{k=b}^{N} k c_k^N(\cdot)\right|\right| \leq \sqrt{\frac{2MT}{b}}. \tag{4.15}$$

Proof. Define $f^b : \mathbb{N} \to \mathbb{R}$ by $f^b(k) := k \wedge b$, and $F^b : \ell_1 \to \mathbb{R}$ by $F^b(\eta) := \langle f^b, \eta\rangle$. We will bound $G^N F^b(\eta)$. Then,

$$k, \ell \geq b \quad \Rightarrow \quad f^b(k+\ell) - f^b(k) - f^b(\ell) = -b,$$

$$\forall k, \ell \in \mathbb{N}, \quad -b \leq f^b(k+\ell) - f^b(k) - f^b(\ell) \leq 0.$$

Therefore

$$\sum_{k,\ell=1}^{N} \Big[f^b(k+\ell) - f^b(k) - f^b(\ell)\Big] k\ell\eta(k)\eta(\ell) \leq -b\left(\sum_{k=b}^{N} k\eta(k)\right)^2. \tag{4.16}$$

Now, taking $f = f^b$ in (4.7), and using (4.16),

$$G^N F^b\Big(c^N(t)\Big) \leq -\frac{b}{2}\left(\sum_{k=b}^{N} k c_k^N(t)\right)^2 + \frac{b}{N}\langle c^N(t), k^2\rangle. \tag{4.17}$$

Note that we ignore the contribution to $G^N F^b(\eta)$ from freezing, since this is negative. Then, since M^{N,F^b} is a martingale, from (4.6) and (4.17) we obtain

$$\begin{aligned}\frac{b}{2}\left|\left|\left(\sum_{k=b}^{N} k c_k^N(\cdot)\right)^2\right|\right| &\leq \mathbb{E}\Big[F^b\Big(c^N(0)\Big)\Big] - \mathbb{E}\Big[F^b\Big(c^N(T)\Big)\Big] + \frac{b}{N}\Big|\Big|\langle c^N(\cdot), k^2\rangle\Big|\Big| \\ &\leq \mathbb{E}\Big[\langle c^N(0), k\rangle\Big] + \frac{b}{N}\Big|\Big|\langle c^N(\cdot), k^2\rangle\Big|\Big|.\end{aligned}$$

Lemma 4.5 shows that this final term vanishes as $N \to \infty$. We then apply Cauchy–Schwarz twice to conclude

$$\limsup_{N\to\infty} \left\| \sum_{k=b}^{N} k c_k^N(\cdot) \right\|^2 \le T \limsup_{N\to\infty} \left\| \left(\sum_{k=b}^{N} k c_k^N(\cdot) \right)^2 \right\| \le \frac{2MT}{b}.$$

The required result (4.15) follows. □

For any integer $b \ge 1$, we now consider a truncated and modified generator G_b defined by

$$G_b F(\eta) = \frac{1}{2} \sum_{k,\ell=1}^{b-1} [f(k+\ell) - f(k) - f(\ell)] k\ell \eta(k)\eta(\ell),$$

for $F(\eta) = \langle f, \eta \rangle$ where $f : \mathbb{N} \to \mathbb{R}$ is bounded. We can compare $G_b F(c^N(t))$ and the original generator $G^N F(c^N(t))$, as defined in (4.7).

Recall (from the conditions of Theorem 4.2, which we are proving) that (v^N) satisfies $v^N(0) \xrightarrow{d} v(0)$ in ℓ_1, and thus there exists $M < \infty$ such that $||v^N(0)||_1 = \langle c^N(0), k \rangle \le M$ for all N. Thus by Lemma 4.3, we have $\langle c^N(t), k \rangle \le M$ for all $t \ge 0$. Then, for $N \ge b$ and $t \in [0, T]$,

$$\begin{aligned}
&\left| G^N F\Big(c^N(t)\Big) - G_b F\Big(c^N(t)\Big) \right| \\
&\le \frac{1}{2} \sum_{\substack{k,\ell=1 \\ k\vee\ell\ge b}}^{N} \Big| f(k+\ell) - f(k) - f(\ell) \Big| k\ell c_k^N(t) c_\ell^N(t) \\
&\quad + \frac{1}{N} \sum_{k=1}^{N} \Big| f(2k) - 2f(k) \Big| k^2 c_k^N(t) + \lambda(N) \sum_{k=1}^{N} \Big| f(k) \Big| k c_k^N(t) \\
&\le \sup_{k\in\mathbb{N}} |f(k)| \left[3\langle c^N(t), k \rangle \left(\sum_{k\ge b}^{N} k c_k^N(t) \right) + \frac{3}{N} \langle c^N(t), k^2 \rangle + \lambda(N) \langle c^N(t), k \rangle \right] \\
&\le \sup_{k\in\mathbb{N}} |f(k)| \left[3M \left(\sum_{k\ge b}^{N} k c_k^N(t) \right) + \frac{3}{N} \langle c^N(t), k^2 \rangle + M\lambda(N) \right].
\end{aligned}$$

Therefore, recalling $\lambda(N) \ll 1$, and applying Lemmas 4.5 and 4.6,

$$\limsup_{N\to\infty} \left\| G^N F\Big(c^N(\cdot)\Big) - G_b F\Big(c^N(\cdot)\Big) \right\| \le 3M \sqrt{\frac{2MT}{b}} \sup_{k\in\mathbb{N}} |f(k)|. \tag{4.18}$$

Since $c^N \to c$ uniformly $\mathbb{P}$-a.s., we also have $G_bF(c^N(\cdot)) \to G_bF(c(\cdot))$ uniformly on $[0,T]$, and so

$$\lim_{N\to\infty} ||G_bF(c^N) - G_bF(c)|| = 0. \tag{4.19}$$

Recall (4.12), which controls the behaviour of the norm $||\cdot||$ under integration. Using this, from (4.18), (4.19), we have

$$\limsup_{N\to\infty} \left|\left| \int_0^\cdot G^N F(c^N(s))\mathrm{d}s - \int_0^\cdot G_bF(c(s))\mathrm{d}s \right|\right| \le \sqrt{\frac{18M^3T^3}{b}} \sup_{k\in\mathbb{N}} |f(k)|. \tag{4.20}$$

Recall from Section 4.1.3, the definition of $M^{N,F}$, the martingale part of $F(c^N(\cdot))$, and let $N \to \infty$ in all the terms on the RHS of (4.6). First, we show that $||M^{N,F}||$ vanishes as $N \to \infty$. Here, again we follow the method of Merle and Normand [51] closely. Any jump of $M^{N,F}$ has size at most $\frac{3}{N}\sup_{k\in\mathbb{N}}|f(k)|$, and as in (4.10), such jumps occur at a rate bounded by

$$\lambda(N)N\langle c^N(0), k\rangle + N\langle c^N(0), k\rangle^2 \le M\lambda(N)N + M^2N.$$

Thus we can bound the expectation of the quadratic variation $[M^{N,F}]_T$, as

$$\limsup_{N\to\infty} \mathbb{E}\Big[[M^{N,F}]_T\Big] \le T \limsup_{N\to\infty}\left(\frac{3}{N}\sup_{k\in\mathbb{N}}|f(k)|\right)^2 \Big[M\lambda(N)N + M^2N\Big] = 0.$$

So by Doob's inequality, as $N \to \infty$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} (M_t^{N,F})^2\right] \le 4\mathbb{E}\Big[(M_T^{N,F})^2\Big] = 4\mathbb{E}\Big[[M^{N,F}]_T\Big] \to 0.$$

By Cauchy–Schwarz, $\mathbb{E}\left[\sup_{t\in[0,T]}\left|M_t^{N,F}\right|\right] \to 0$, from which we conclude

$$||M^{N,F}|| \to 0. \tag{4.21}$$

The assumption that $c^N \to c$ uniformly on $[0,T]$ implies $F(c^N) \to F(c)$ in $||\cdot||$. So we let $N \to \infty$ in (4.6) and use (4.20) and (4.21) to obtain

$$\left|\left|F(c(\cdot)) - F(c(0)) - \int_0^{\cdot} \frac{1}{2} \sum_{k,\ell=1}^{b-1} [f(k+\ell) - f(k) - f(\ell)] k\ell c_k(t) c_\ell(t)\right|\right| \leq \alpha_b,$$

where $\alpha_b := \sqrt{\frac{18M^3T^3}{b}} \sup_{k\in\mathbb{N}} |f(k)|$. We may take $b \to \infty$, and observe that c is $\mathbb{P}$-a.s. continuous, so we find that c solves (4.4) for all $t \in [0,T]$ as required.

4.2.2 Convergence of (Φ^N) and (v^N)

As Merle and Normand observe in Section 2.3 of [51], the convergence $c^N \to c$ in $\mathbb{D}([0,\infty),\ell_1)$ does not imply the convergence $\Phi^N \to \Phi$ in $\mathbb{D}([0,\infty))$ immediately. However, convergence of Φ^N can be established quickly as follows.

From the monotone convergence theorem,

$$\lim_{b\to\infty} \left|\left|\Phi(\cdot) - \sum_{k=1}^{b-1} kc_k(\cdot)\right|\right| = 0. \tag{4.22}$$

For the discrete processes, we can rewrite (4.15) as

$$\lim_{b\to\infty} \limsup_{N\to\infty} \left|\left|\Phi^N(\cdot) - \sum_{k=1}^{b-1} kc_k^N(\cdot)\right|\right| = 0. \tag{4.23}$$

Finally, since almost surely $c^N \to c$ uniformly with respect to ℓ_1, we have

$$\lim_{N\to\infty} \left|\left|\sum_{k=1}^{b-1} kc_k^N(\cdot) - \sum_{k=1}^{b-1} kc_k(\cdot)\right|\right| = 0 \quad \forall b \geq 1. \tag{4.24}$$

Adding (4.22), (4.23) and (4.24), and letting $b \to \infty$, we obtain

$$\lim_{N\to\infty} \left|\left|\Phi^N(\cdot) - \Phi(\cdot)\right|\right| = 0. \tag{4.25}$$

It remains to show that Φ^N is uniformly close to Φ in probability. We observe that Φ^N is monotone decreasing, and Φ is monotone decreasing and uniformly continuous, by

Theorem 4.1. That is, for every $\delta \in (0, T]$ we have a constant $C(\delta)$ such that whenever $|s-t| \le \delta$ for $s, t \in [0, T]$, then $|\Phi(s) - \Phi(t)| \le C(\delta)$, and we may assume that $C(\delta) \to 0$ as $\delta \to 0$.

Choose some $\delta < T/2$. Then suppose $\exists t \in [0, T/2]$ for which $\Phi(t) \ge \Phi^N(t) + 2C(\delta)$. Then

$$\int_t^{t+\delta} |\Phi^N(s) - \Phi(s)| \mathrm{d}s \ge \int_t^{t+\delta} \Big[\Phi(t) - C(\delta) - \Phi^N(t)\Big] \mathrm{d}s \ge \delta C(\delta).$$

Therefore

$$\limsup_{N \to 0} \mathbb{P}\Bigg(\sup_{t \in [0, T/2]} \Big[\Phi(t) - \Phi^N(t)\Big] \ge 2C(\delta) \Bigg) \le \frac{\limsup_{N \to \infty} \Big|\Big|\Phi^N(\cdot) - \Phi(\cdot)\Big|\Big|}{\delta C(\delta)} = 0. \tag{4.26}$$

Similarly, if $\exists t \in [\delta, T/2]$ for which $\Phi^N(t) \ge \Phi(t) + 2C(\delta)$, then

$$\int_{t-\delta}^{t} |\Phi^N(s) - \Phi(s)| \mathrm{d}s \ge \int_{t-\delta}^{t} \Big[\Phi^N(t) - (\Phi(t) + C(\delta)\Big] \mathrm{d}s \ge \delta C(\delta).$$

and so

$$\limsup_{N \to 0} \mathbb{P}\Bigg(\sup_{t \in [\delta, T/2]} \Big[\Phi^N(t) - \Phi(t)\Big] \ge 2C(\delta) \Bigg) \le \frac{\limsup_{N \to \infty} \Big|\Big|\Phi^N(\cdot) - \Phi(\cdot)\Big|\Big|}{\delta C(\delta)} = 0. \tag{4.27}$$

Finally, observe that

$$\sup_{t \in [0,\delta]} \Big[\Phi^N(t) - \Phi(t)\Big] \le \Big[\Phi^N(0) - \Phi(\delta)\Big].$$

But, by assumption, $\Phi^N(0) \xrightarrow{\mathbb{P}} \Phi(0)$ as $N \to \infty$, and $\Phi(\delta) \to \Phi(0)$ as $\delta \to 0$. So, combining with (4.26) and (4.27), we obtain

$$\sup_{t \in [0, T/2]} \Big|\Phi^N(t) - \Phi(t)\Big| \xrightarrow{\mathbb{P}} 0, \tag{4.28}$$

as $N \to \infty$. Since $T > 0$ was arbitrary, we may replace $T/2$ by T in the statement of (4.28).

Finally, we want to show convergence of (v^N). We could appeal to Scheffé's Lemma, but as it is short, we give an argument from first principles. Note that for any $b \geq 1$, and $t \geq 0$,

$$||v^N(t) - v(t)||_1 \leq b||c^N(t) - c(t)||_1 + \left|\Phi^N(t) - \sum_{\ell=1}^{b-1} v_\ell^N(t)\right| + \left|\Phi(t) - \sum_{\ell=1}^{b-1} v_\ell(t)\right|. \quad (4.29)$$

As $b \to \infty$, the third term of (4.29) vanishes uniformly on $t \in [0, T]$. As $N \to \infty$, the second term approaches the third term uniformly in probability by the convergence of c^N to c, and by (4.28). Similarly, as $N \to \infty$ the first term vanishes uniformly in probability.

Therefore

$$\sup_{t \in [0,T]} ||v^N(t) - v(t)||_1 \to 0, \quad \mathbb{P} - \text{a.s.}$$

and the proof of Theorem 4.2 is complete.

Chapter 5

Frozen percolation with k types

5.1 Discrete models and a limit object

We consider frozen percolation, started from an inhomogeneous random graph with k types, and seek to understand the asymptotic behaviour of the proportion of alive vertices of each type. The evolution in time of these asymptotic proportions will be described by the solution to an $\mathbb{R}^k$-valued differential equation, which we call a *frozen percolation type flow.*

5.1.1 Multitype frozen percolation processes

Definition 5.1. We assume throughout that a lightning rate $\lambda : \mathbb{N} \to \mathbb{R}_+$ is given, satisfying $1/N \ll \lambda(N) \ll 1$, exactly as in the original mean-field frozen percolation model. As in Chapter 3, we also fix an integer k, and we will consider kernels and graphs with k types throughout this chapter. Let p be a (possibly random) element of $\mathbb{N}_0^k$, and κ a non-negative kernel. We define the *multitype frozen percolation process with index N*, denoted $\left(\mathcal{G}^{N,p,\kappa,\lambda(N)}(t)\right)_{t\geq 0}$, as follows. We set $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$ to be a copy of $G^N(p,\kappa)$, an inhomogeneous random graph with k types, as in Definition 3.2.

The set of vertices is $[\sum p_i]$, and the type each vertex is assigned in the initial configuration remains fixed as time advances. Then we run the frozen percolation dynamics with lightning rate $\lambda(N)$. That is, we declare all vertices to be alive initially, and we add edges at rate $1/N$ independently between any pair of alive vertices that are not already connected by an edge. Independently, each vertex is struck by lightning at rate $\lambda(N)$, and when this happens, all the vertices in its component are declared frozen. With these dynamics, we write $\mathcal{G}^{N,p,\kappa,\lambda(N)}(t)$ for the graph of alive vertices, with their types, at time t.

We will assume that a kernel κ and a $\mathbb{N}_0^k$-valued sequence p^N is given. We will consider the corresponding sequence of processes $(\mathcal{G}^{N,p^N,\kappa,\lambda(N)})$. Sometimes we will suppress notational dependence on p^N, κ and $\lambda(N)$.

For every time $t \geq 0$, and $i \in [k]$, we define $\pi_i^N(t) := \frac{1}{N}\#\{\text{alive vertices of type } i \text{ at time } t\}$, and $\Phi^N(t) := \sum_{i=1}^k \pi_i^N(t)$, the total proportion of alive vertices at time t. We now state a simple consequence of the definition which we will use repeatedly. Recall that $\mathbf{1}$ is the $k \times k$ matrix for which every entry is 1.

It will be important throughout to distinguish between various types of conditioning. To this end, we let $(\mathcal{F}^N(t))_{t\geq 0}$ be the natural filtration of $(\pi^N(t))$.

Proposition 5.2. Let $\hat{\mathcal{G}}^N(t)$ be obtained from $\mathcal{G}^N(t)$ by relabelling the alive vertices with $\{1,\dots,N\Phi^N(t)\}$, uniformly at random. Then, conditional on $\mathcal{F}^N(t)$, $\hat{\mathcal{G}}^N(t)$ has the same distribution as $G^N(N\pi^N(t), \kappa + t\mathbf{1})$ on the set of graphs with k types.

Proof. To make the argument more clear, we condition on additional information. For now, take $t \geq 0$ fixed. Given the frozen percolation process $\mathcal{G}^{N,p,\kappa,\lambda(N)}$, we define the sigma-algebra $\hat{\mathcal{F}}^N(t)$ generated by $(\pi^N(s), s \in [0,t])$, *and* the types of all vertices assigned at time 0, *and* $\mathcal{A}(t)$, the set of alive vertices at time t. We claim that conditional on $\hat{\mathcal{F}}^N(t)$, $\hat{\mathcal{G}}^N(t)$ has the same distribution as $G^N\Big(N\pi^N(t), \kappa + t\mathbf{1}\Big)$ on the set of graphs with k types, which clearly implies the statement of the proposition.

For brevity, we set $P := \sum p_i$. In the definition of a frozen percolation process $\mathcal{G}^{N,p,\kappa,\lambda(N)}$, we can consider the edge-arrival process to be a Poisson point process $\mathcal{E}$ on $\binom{[P]}{2} \times [0,\infty)$, with intensity given by the product of counting measure and Lebesgue measure, scaled by $\frac{1}{N}$. We also consider the lightning process to be a Poisson point process $\mathcal{L}$ on $[P] \times [0,\infty)$, with intensity given by the product of counting measure and Lebesgue measure, scaled by $\lambda(N)$. In particular, we take $\mathcal{E}$ and $\mathcal{L}$ to be independent.

Given $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$, we can recover the whole frozen percolation process $\mathcal{G}^{N,p,\kappa,\lambda(N)}$ on $[0,\infty)$ using $\mathcal{E}$ and $\mathcal{L}$ in the natural way, ignoring points in $\mathcal{E}$ corresponding to edges which are already present, and points in both $\mathcal{E}$ and $\mathcal{L}$ that correspond to already-frozen vertices. In particular, the evolution of $\mathcal{G}^{N,p,\kappa,\lambda(N)}$ on $[0,t]$ is independent of the restrictions of $\mathcal{E}$ and $\mathcal{L}$ to $\binom{[P]}{2}\times[t,\infty]$ and $[P]\times[t,\infty]$, respectively.

Let type be the (random) type function $[P]\to[k]$ of the initially-alive vertices. Let $\mathbf{t}$ be any function $[P]\to[k]$ satisfying $|\mathbf{t}^{-1}(i)|=p_i$ for all $i\in[k]$, and let $A\subseteq[P]$. Then the event $B=\{\text{type}=\mathbf{t},\,\mathcal{A}(t)=A\}$ depends precisely on

1. the types in $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$;
2. the restriction of the edge set of $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$ to $\binom{[P]}{2}\backslash\binom{[A]}{2}$;
3. the restriction of $\mathcal{E}$ to $\left(\binom{[P]}{2}\backslash\binom{[A]}{2}\right)\times[0,t]$;
4. the restriction of $\mathcal{L}$ to $[P]\times[0,t]$.

Furthermore, conditional on $\{\text{type}=\mathbf{t},\,\mathcal{A}(t)=A\}$, the restriction of π^N to $[0,t]$ depends only on conditions 1 to 4 as well. Therefore, B, is independent of the restriction of the edge set of $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$ to $\binom{[A]}{2}$, and the restriction of $\mathcal{E}$ to $\binom{[A]}{2}\times[0,\infty)$, and thus so is π^N restricted to $[0,t]$, conditional on B.

It follows that, conditional on $\hat{\mathcal{F}}^N(t)$, the vertex set of $\mathcal{G}^{N,p,\kappa,\lambda(N)}(t)$ is $\mathcal{A}(t)$, and the types are given by the restriction of type to $\mathcal{A}(t)$. Since $|\mathcal{A}(t)|=||\pi^N(t)||_1$, the distribution of the latter is the uniform distribution among functions $f:\mathcal{A}(t)\to[k]$ satisfying $|f^{-1}|=N\pi_i^N(t)$ for all $i\in[k]$. With this conditioning, the presence of an edge between vertices $x,y\in\mathcal{A}(t)$ depends only on the presence of an edge between x,y in $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$, and the restriction of $\mathcal{E}$ to $\{x,y\}\times[0,t]$, and so has probability

$$1-\exp\left(-\frac{\kappa_{\text{type}(x),\text{type}(y)}}{N}\right)\cdot\exp(-t/N).$$

Furthermore, since conditional on types, edges in $\mathcal{G}^{N,p,\kappa,\lambda(N)}(0)$ are independent, and the restrictions of $\mathcal{E}$ to different first arguments are independent, it follows that different edges between vertices in $\mathcal{A}(t)$ are independent also. Therefore, conditional on $\hat{\mathcal{F}}^N(t)$, after uniformly random relabelling of the vertices, $\mathcal{G}^{N,p,\kappa,\lambda(N)}(t)$ has precisely the distribution of $G^N(N\pi^N(t), \kappa + t\mathbf{1})$ on the set of graphs on $[P]$ with k types, as required. □

Remark. In the definition of $G^N(p,\kappa)$ in Chapter 3, we demanded that each edge be present with probability $1 - \exp(-\kappa_{i,j}/N)$ rather than $\kappa_{i,j}/N$. This choice makes the statement of this proposition simpler.

The aim of this chapter is to give a description of the limit of the processes π^N in $\mathbb{D}^k([0,\infty))$, when p^N/N converges in distribution to a constant vector.

5.1.2 Frozen percolation type flows

Definition 5.3. Given a non-negative kernel κ, motivated by Proposition 5.2, we define a time-dependent kernel for each time t in terms of κ by

$$\kappa(t) = \kappa + t\mathbf{1}. \tag{5.1}$$

Recall from Definition 3.5 that $\rho(A)$ is the Perron root of a positive matrix A, and $\mu(A)$ is the corresponding principal left-eigenvector, normalised so that $\sum \mu_i(A) = 1$. Also recall from (3.1) the definitions

$$\Pi_1 := \left\{ \pi \in \mathbb{R}^k_{\geq 0} \,:\, \sum_{i\in[k]} \pi_i = 1 \right\}, \quad \Pi_{\leq 1} := \left\{ \pi \in \mathbb{R}^k_{\geq 0} \,:\, \sum_{i\in[k]} \pi_i \leq 1 \right\}.$$

Definition 5.4. We say $\pi : [0,\infty) \to \mathbb{R}^k_{\geq 0}\backslash\{0\}$ is a *frozen percolation type flow* with initial non-negative kernel κ and positive initial measure $\pi(0) \in \Pi_{\leq 1}$ if π is continuous and there exists some *critical time* $t_c \geq 0$ such that:

$$\pi(t) = \pi(0), \quad t \leq t_c, \tag{5.2}$$

$$\rho(\kappa(t) \circ \pi(t)) = 1, \quad t \geq t_c, \tag{5.3}$$

$$\frac{d}{dt}\pi(t) = -\mu(\kappa(t) \circ \pi(t))\phi(t), \quad t > t_c. \tag{5.4}$$

where $\phi : (t_c, \infty) \to \mathbb{R}_+$ is continuous. In addition, we define $\Phi(t) = \sum_{i=1}^k \pi_i(t)$.

Our main results are:

THEOREM 5.5. We consider $\kappa \in \mathbb{R}^{k\times k}_{\geq 0}$ and positive $\pi(0) \in \Pi_1$. Assume that at least one of the following holds:

- κ is a positive kernel, and $\rho(\kappa \circ \pi(0)) \leq 1$;
- $\rho(\kappa \circ \pi(0)) < 1$.

Then there exists a unique frozen percolation type flow with initial kernel κ started from distribution $\pi(0)$.

Note. Assuming that at least one of the two above conditions holds ensures that $\kappa(t_c)$ is positive, which in turn ensures that $\mu(\kappa(t_c) \circ \pi(t_c))$ is well-defined without investigating further conditions about irreducibility of κ.

It is an easy consequence of (5.1) and (5.3) (that we will explain in Section 5.4) that $\Phi(t) \to 0$ as $t \to \infty$. We can show furthermore that any frozen percolation type flow has a limiting *proportion* of types.

THEOREM 5.6. For any frozen percolation type flow with positive $\pi(0)$, $\lim_{t\to\infty} \frac{\pi(t)}{\Phi(t)}$ exists, and is positive.

The main theorem, and the motivation for considering frozen percolation type flows, is the following.

THEOREM 5.7. Fix κ and $\pi(0)$ satisfying the conditions of Theorem 5.5, and $\lambda : \mathbb{N} \to \mathbb{R}_+$ satisfying the usual critical scaling (4.3). Consider a family of multitype frozen percolation processes $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}$, such that $p^N/N \xrightarrow{d} \pi(0)$. Then the following process convergence holds:

$$\pi^N(\cdot) \to \pi(\cdot),$$

in distribution with respect to the uniform topology on $\mathbb{D}^k([0,T])$ as $N \to \infty$ for each $T < \infty$, where π is the unique frozen percolation type flow with initial kernel κ started from distribution $\pi(0)$.

Remark. The multitype frozen percolation processes exhibit self-organised criticality in the form of power-law tails for component sizes since they are special cases of Ráth's original mean-field frozen percolation model. However, (5.3) gives an alternative and slightly simpler interpretation of criticality in this context.

Outline of argument

We prove the uniqueness result of Theorem 5.5 in Section 5.2. We show that for any solution to (5.4), there is an associated solution to the Smoluchowski equations (4.1), with the same control function $\Phi(\cdot)$. We can then lift Theorem 4.1's result about uniqueness of solutions to these equations to the setting we require.

We prove Theorem 5.7 in Section 5.3, from which the existence result of Theorem 5.5 follows. We show tightness of the family of processes (π^N) in $\mathbb{D}^k([0,T])$. We know from Theorem 4.2 that Φ^N converges to Φ, and we will use this to control the behaviour of the weak limits of (π^N). We show that any weak limit features critical graphs after the gelation time, as otherwise Φ is either discontinuous or locally constant, which contradicts Theorem 4.1. Checking that weak limits π satisfy (5.4) involves a careful estimate of the number of vertices lost to freezing, using the results obtained in Chapter 3. Finally, we prove Theorem 5.6 in the short Section 5.4.

5.1.3 Motivation - ages in forest fires

We explain briefly how a novel interpretation of the mean-field forest fire process introduced in Section 1.3.2 motivates the multitype frozen percolation processes studied in this chapter.

Consider a mean-field forest fire process on $[N]$, started from the empty graph. At each time $t \geq 0$, some vertices may have been burned some number of times. We associate with each vertex $i \in [N]$ the process

$$s_i(t) = \max\{s \in [0,t] : \text{component containing } i \text{ struck by lightning at time } s\}. \quad (5.5)$$

Here we take $\max \varnothing = 0$. That is, $s_i(t)$ tracks the time at which vertex i was most recently involved in a fire. Now, for some fixed time $t \geq 0$ we consider the structure of the graph, conditional on the sequence $(s_1(t), \dots, s_N(t))$. This conditioning implies that each vertex $i \in [N]$ was not struck by lightning on the interval $(s_i(t), t]$. In fact, we can make the following stronger statement, which we justify at the end of this section.

Proposition 5.8. Conditional on $(s_1(t), \dots, s_N(t))$, the probability of an edge between vertices i and j is

$$1 - \exp\left[-\frac{t - s_i(t) \vee s_j(t)}{N}\right]. \tag{5.6}$$

Furthermore, the events that distinct edges in $[N]^{(2)}$ are present are independent under this conditioning.

Remark. In other words, conditional on this sequence of burning times, the graph at time t is an inhomogeneous random graph, with type-space parameterised by $[0, T]$.

We then define the following process, recording the empirical distribution of burning times

$$\pi^N(t, \cdot) := \frac{1}{N} \sum_{i \in [N]} \delta_{s_i(t)}(\cdot). \tag{5.7}$$

We conjecture the existence and properties of a limit distribution $\pi(t) = \lim_{N \to \infty} \pi^N(t)$. This $\pi(t)$ has a Dirac delta weight at zero, corresponding to the probability that a given vertex has never been burnt before time t. Given $\pi(t, \cdot)$, we can describe the local limit of the graph of the forest fire process at time t, as a multitype branching process tree with type space $[0, t]$. The root has type distributed as $\pi(t, \cdot)$, and thereafter, any vertex with type s has offspring on the type space distributed according to a Poisson random measure with intensity

$$\Psi(t, s, u) := \pi(t, u)(t - s \vee u).$$

For any post-gelation time $t \geq T_g$ we expect the parameters of the inhomogeneous random graph to be critical. That is, there exists a distribution $\mu(t, \cdot)$ on $[0, t]$ for which

$$\int_0^t \mu(t, s)\Psi(t, s, u)\mathrm{d}s = \mu(t, u), \quad u \in [0, t], \quad \int_0^t \mu(t, s)\mathrm{d}s = 1,$$

and which we view as a left-1-eigenfunction for the operator Ψ. Again $\mu(t, \cdot)$ has a Dirac mass at zero.

In addition, for $t \in (T_g, T]$, we would expect the burning time distribution $\pi(t, \cdot)$ to evolve according to the following differential equation, analogous to (5.4),

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t,s) = -\phi(t)\mu(t,s) + \phi(t)\delta_t(s), \quad t \in [T_g, T],\, s \in [0,T]. \tag{5.8}$$

It can be shown that the stationary solution (1.23) to the modified Smoluchowski equations (1.22) corresponds to a burning-time distribution on $(-\infty, 0]$. So a result about existence and uniqueness of solutions to (5.8) analogous to Theorem 5.5 offers a new approach to outstanding questions of existence and uniqueness for the modified Smoluchowski equations, for some initial conditions.

However, there are substantial analytic technicalities involved in working with a continuous (and possibly infinite) type-space, hence the motivation for considering the simpler but related model of frozen percolation with a finite number of types in this chapter. This approach to the forest fire by considering ages in a continuous typespace is the subject of ongoing work with Crane and Ráth.

Proof of Proposition 5.8. As in the proof of Proposition 5.2, let $\mathcal{E}$ be a Poisson point process on $\binom{[N]}{2} \times [0,\infty)$, and $\mathcal{L}$ be an independent Poisson point process on $[N] \times [0,\infty)$. As in the frozen percolation process, we can construct the mean-field forest fire process on $[N]$ from a realisation of $(\mathcal{E}, \mathcal{L})$. Indeed in this setting, the initial configuration (recall, the empty graph on $[N]$) is deterministic, and so this pair of independent PPPs is enough to define the forest fire process. (For definiteness, we say that if an edge arrives at the same time as a lightning strike in $(\mathcal{E}, \mathcal{L})$, we add the edge before considering the effect of the lightning. Obviously the probability that this happens at some time in $[0, T]$ is zero for all $T > 0$.)

We claim the following equality of events, from which the statement of the proposition will follow rapidly.

Lemma 5.9. Take $\sigma \in [0,t]^N$, and recall the definition of $s(t) = (s_1(t), \dots, s_N(t))$ from (5.5). Then

$$\{s(t) = \sigma\} = \bigcap_{i \in [N]} \{s_i(\sigma_i) = \sigma_i\} \cap \left\{ \mathcal{L}\left(\bigcup_{i \in [N]} \{i\} \times (\sigma_i, t] \right) = 0 \right\}$$
$$\cap \left\{ \mathcal{E}\left(\bigcup_{i \neq j} \{i,j\} \times (\sigma_i \wedge \sigma_j, \sigma_i \vee \sigma_j] \right) = 0 \right\}. \tag{5.9}$$

Proof. For brevity, we let $\mathcal{B}(s(t),\sigma)$ be the event on the RHS of (5.9). We start by showing that $\{s(t) = \sigma\} \subseteq \mathcal{B}(s(t),\sigma)$. It is immediately clear that $s(t) = \sigma$ implies $s_i(\sigma_i) = \sigma_i$ and $\mathcal{L}(\{i\} \times (\sigma_i, t]) = 0$ for each $i \in [N]$. It remains to show that the final event in the intersection which defines $\mathcal{B}(s(t),\sigma)$ holds.

Suppose $s(t) = \sigma$, but there exist $i \neq j \in [N]$ for which $\mathcal{E}(\{i,j\} \times (\sigma_i \wedge \sigma_j, \sigma_i \vee \sigma_j])$ is positive. If $\sigma_i = \sigma_j$ then $(\sigma_i \wedge \sigma_j, \sigma_i \vee \sigma_j] = \varnothing$, which contradicts this positivity, and so without loss of generality we assume $\sigma_i < \sigma_j$. Now, since i is not struck by lightning after time $s_i(t) = \sigma_i$, and $\mathcal{E}(\{i,j\} \times (\sigma_i, \sigma_j]) > 0$, the edge formed between i and j has not been deleted as a result of lightning until time σ_j. Therefore, when j is involved in a fire at time σ_j, so is i, which means $s_i(t) \geq \sigma_j$, a contradiction. We have shown that $\{s(t) = \sigma\} \subseteq \mathcal{B}(s(t),\sigma)$.

We now assume that $\mathcal{B}(s(t),\sigma)$ holds. From $\cap\{s_i(\sigma_i) = \sigma_i\}$, we obtain $s(t) \geq \sigma$. Now suppose for contradiction that there is some $i \in [N]$ such that $s_i(t) > \sigma_i$, and we choose such an i with σ_i maximal. Let j be the vertex struck by lightning at time $s_i(t)$, which causes the fire affecting i. If there is more than one such j, choose any one. (Note the probability that two vertices are struck by lightning simultaneously is zero.) Since $\mathcal{L}(\{j\} \times (\sigma_j, t]) = 0$, we must have $\sigma_j \geq s_i(t) > \sigma_i$. So by maximality in the choice of i, we must have $\sigma_j = s_j(t)$.

Therefore, there is a path of distinct vertices $i = i_0, i_1, \dots, i_k = j$ such that each edge $\{i_m, i_{m+1}\}$ is present at time $s_i(t)$. Since we have $s_i(t) > \sigma_i$, and $s_j(t) = \sigma_j$, there must exist $0 \leq m \leq k-1$ for which $s_{i_m}(t) > \sigma_{i_m}$ and $s_{i_{m+1}}(t) = \sigma_{i_{m+1}}$. Now, again by maximality in the choice of i, we must have

$$\sigma_{i_m} \le \sigma_i < s_i(t), \quad \text{and} \quad \sigma_{i_{m+1}} = s_{i_{m+1}}(t) \ge s_i(t),$$

where the final inequality holds since i_{m+1} is in a fire at time $s_i(t)$, by construction. In particular, $\sigma_{i_m} < \sigma_{i_{m+1}}$.

The edge $\{i_m, i_{m+1}\}$ is present at time $s_i(t)$, but we have assumed $s_{i_m}(\sigma_{i_m}) = \sigma_{i_m}$, and so it is not present at time σ_{i_m}. However,

$$\mathcal{E}\big(\{i_m, i_{m+1}\} \times (\sigma_{i_m}, \sigma_{i_{m+1}}]\big) = 0 \quad \text{implies} \quad \mathcal{E}\big(\{i_m, i_{m+1}\} \times (\sigma_{i_m}, s_i(t)]\big) = 0,$$

so therefore the edge is not added before time $s_i(t)$. This contradiction shows that $s_i(t) = \sigma_i$ for all $i \in [N]$. Therefore we have shown the inclusion relation corresponding to (5.9) in both directions, and the proof of the lemma is complete. □

To finish the proof of Proposition 5.8, note that for all $\sigma \in [0,t]^N$, the event $\mathcal{B}(s(t), \sigma)$ as defined in (5.9) is certainly independent of $\mathcal{E}$ restricted to

$$\bigcup_{i \ne j} \{i,j\} \times (\sigma_i \vee \sigma_j, t].$$

In particular, conditional on $\{s(t) = \sigma\}$, the restrictions of $\mathcal{E}$ to $\{i,j\} \times (\sigma_i \vee \sigma_j, t]$ remain independent across edges $\{i,j\} \in [N]^{(2)}$. Finally, on $\{s(t) = \sigma\}$, an edge is present between i and j at time t precisely if

$$\mathcal{E}(\{i,j\} \times (\sigma_i \wedge \sigma_j, t]) > 0,$$

which from the final term in the intersection on the RHS of (5.9) is, on $\{s(t) = \sigma\}$, equivalent to

$$\mathcal{E}(\{i,j\} \times (\sigma_i \vee \sigma_j, t]) > 0.$$

The probability in (5.6) follows immediately, and with the independence that we have already shown, the proof of Proposition 5.8 is complete. □

5.2 Uniqueness of frozen percolation type flows

In this section, we prove the following proposition.

Proposition 5.10. Consider kernel $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$ and $\pi(0) \in \Pi_{\leq 1}$ satisfying one of the conditions in Theorem 5.5. Suppose there are frozen percolation type flows π, ν, both with initial kernel κ started from distribution $\pi(0)$. Then $\pi = \nu$.

The proof proceeds by constructing a solution to the Smoluchowski equations (4.1) from a frozen percolation type flow. We will use Theorem 4.1 to conclude that these are the same for both π and ν, and in particular, the associated Φs are the same.

In the following lemma, we show that every component of $\pi(t)$ stays positive for all finite $t \geq 0$. This natural condition avoids the requirement for an awkward case distinction in the main argument of this section.

Lemma 5.11. Any frozen percolation type flow $(\pi(t))_{t\geq 0}$, with initial kernel $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$ and positive initial distribution $\pi(0) \in \Pi_{\leq 1}$, is positive for all times $t \geq 0$.

Proof. We write $\mu(t)$ as an abbreviation for $\mu(\kappa(t) \circ \pi(t))$. Also recall the definition $\kappa_{\max} := \max_{i,j\in[k]} \kappa_{i,j}$. The result is clear for $t \leq t_c$. Now suppose that

$$T := \inf\{t > t_c \,:\, \exists i \in [k], \pi_i(t) = 0\} < \infty.$$

Observe that $T > t_c$ since $\pi(t_c) = \pi(0)$ and π is continuous. Then consider any $t \in [t_c, T)$. Since $\rho(\kappa(t) \circ \pi(t)) = 1$ and $\kappa(t) \circ \pi(t)$ is positive, the eigenvector $\mu(t)$ is well-defined, and satisfies

$$\mu_i(t) = \pi_i(t) \sum_{j=1}^{k} \mu_j(t) \cdot (\kappa_{j,i} + t).$$

So, since $\pi_i \leq 1$ and $\sum \mu_j = 1$, we have

$$\mu_i(t) \leq \pi_i(t)[\kappa_{\max} + t],$$

So from (5.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) \geq -\pi(t) \cdot \phi(t)[\kappa_{\max} + t].$$

Thus

$$\pi(t) \geq \pi(t_c) \exp\Big(- \int_{t_c}^{t} [\kappa_{\max} + s] \phi(s) \mathrm{d}s \Big)$$
$$\geq \pi(0) \exp(-[\kappa_{\max} + t]),$$

since $\int_{t_c}^{t} \phi(s) \mathrm{d}s = \Phi(t_c) - \Phi(t) \leq 1$. This holds for all $t \in [t_c, T)$, and thus

$$\pi(T) \geq \pi(0) \exp(-[\kappa_{\max} + T]),$$

since π is continuous (because π is a frozen percolation type flow). □

5.2.1 FP flows give solutions to Smoluchowski's equations

Given $\kappa \in \mathbb{R}_{\geq 0}^{k \times k}$ and $\pi \in \Pi_{\leq 1}$, recall from Definition 3.4 the Poisson branching process tree with k types $\Xi^{\pi,\kappa}$. Given a frozen percolation type flow with initial kernel κ and initial measure π, we write $\Xi^{(t)}$ as a shorthand for $\Xi^{\pi(t),\kappa(t)}$.

The motivation for introducing branching processes at this point is that, for a family of processes $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}$ satisfying the conditions of Theorem 5.7, $\Xi^{(t)}$ is the Benjamini–Schramm limit (with k types) of $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}(t)$. In particular, since we are in the subcritical and critical regimes, the distribution of the size of $\Xi^{(t)}$ is the limit of the distribution of the size of a component containing a uniformly-chosen (possibly frozen) vertex in $[N]$.

We define $v_\ell(t) := \mathbb{P}\Big(|\Xi^{(t)}| = \ell\Big)$ for $t \geq 0$, $\ell \geq 1$. We will show that shortly that $(v(t))_{t \geq 0}$ satisfies the Smoluchowski equations (4.1). First, we explain how to treat $\mathbb{P}(|\Xi^{\pi,\kappa}| = \ell)$ as a sum over trees. For use in the rest of this section, for any finite set A we define T_A to be the set of *unrooted*, unordered trees, labelled by A, and we define T_A^ρ to be the set of *rooted*, unordered trees, again labelled by A.

Lemma 5.12. Let $\kappa \in \mathbb{R}_{\geq 0}^{k \times k}$ and $\pi \in \Pi_{\leq 1}$. Then

$$\mathbb{P}(|\Xi^{\pi,\kappa}| = \ell) = \frac{1}{\ell!} \sum_{T \in T_{[\ell]}^\rho} \sum_{\substack{i_1,\dots,i_\ell \\ \in [k]}} \Bigg[\prod_{(m,n) \in E(T)} \kappa_{i_m,i_n} \Bigg] \prod_{m=1}^{\ell} \pi_{i_m} \exp\Bigg(- \sum_{j=1}^{k} \kappa_{i_m,j} \pi_j \Bigg). \quad (5.10)$$

Proof. We define $\Delta_i := \sum_{j=1}^{k} \kappa_{i,j}\pi_j$ as we will use this expression frequently. Recall from Definitions 3.3 and 3.4 that Δ_i is the expected number of offspring (of all types) of a type i parent in $\Xi^{\kappa,\pi}$.

To simplify some of the expressions to follow shortly, we will use a slightly different construction of an inhomogeneous random graph with index N, where the set of vertices is also random, corresponding to the type sub-distribution π. More formally, we define a random variable

$$X_1 = \begin{cases} i & \text{with probability} \quad \pi_i, \quad i \in [k] \\ 0 & \text{with probability} \quad 1 - \Phi := 1 - \sum_{i=1}^{k} \pi_i, \end{cases} \tag{5.11}$$

and let $X_2, \dots, X_N$ be IID copies of X_1. We then construct a random graph $\tilde{G}^N(\pi, \kappa)$, conditional on $(X_1, \dots, X_N)$ as follows. The vertex set is $M := \{m \in [N] : X_m \neq 0\}$, and the type of any i in the vertex set is X_i. Then, (as in the original Definition 3.2 of $G^N(p, \kappa)$) each edge $ij \in M^{(2)}$ is present with probability $1 - \exp\Big(-\kappa_{X_i,X_j}/N\Big)$, independently of all other pairs.

Shortly, we will consider the quantities

$$\bar{p}_i^{N,\ell} := \#\{m \in [\ell+1, N] : X_m = i\}, \quad i \in [k], \quad 0 \le \ell \le N-1, \tag{5.12}$$

associated with a realisation of $\tilde{G}^N(\pi, \kappa)$. We will consider local limits in $\bar{G}^N(\pi, \kappa)$. In this setting, we say that $|C(1)|$, the size of the component containing 1, is zero if $X_1 = 0$, that is if 1 is not in the vertex set of $\tilde{G}^N(\pi, \kappa)$.

For any $\ell \le N$, we have

$$\begin{aligned}
\mathbb{P}\Big(|C(1)| = \ell \text{ in } \tilde{G}^N(\pi,\kappa)\Big) &= \binom{N-1}{\ell-1} \mathbb{P}\Big(C(1) = [\ell] \text{ in } \tilde{G}^N(\pi,\kappa)\Big) \\
&= \binom{N-1}{\ell-1} \sum_{T \in T_{[\ell]}} \sum_{\substack{i_1,\dots,i_\ell \\ \in [k]}} \prod_{m \in [\ell]} \pi_{i_m} \prod_{(m,n) \in E(T)} (1 - \exp(-\kappa_{i_m,i_n}/N)) \\
&\qquad \prod_{\substack{(m,n) \in [\ell]^{(2)} \\ (m,n) \notin E(T)}} \exp(-\kappa_{i_m,i_n}/N)\, \mathbb{E}\left[\prod_{m=1}^{\ell} \prod_{j=1}^{k} \exp\left(-\frac{\kappa_{i_m,j}\bar{p}_j^{N,\ell}}{N}\right)\right]
\end{aligned}$$

$$+\,\mathbb{P}\Big(|C(1)|=\ell \text{ and } C(1) \text{ includes a cycle in } \tilde{G}^N(\pi,\kappa)\Big). \tag{5.13}$$

In the first two lines, these products govern, respectively, the probabilities that the vertices in $[\ell]$ are present and have types $(i_1,\dots,i_\ell)$; that the correct edges are present within $[\ell]$; that the correct non-edges are present within $[\ell]$; and the expectation (over random variables $(\bar{p}_j^{N,\ell})_{j\in[k]}$) gives the probability there are no edges between $[\ell]$ and $[N]\backslash[\ell]$, given the types of vertices $[\ell]$.

The following convergence results hold immediately, for all $i_1,\dots,i_\ell\in[k]$,

$$\lim_{N\to\infty}\prod_{\substack{(m,n)\in[\ell]^{(2)}\\(m,n)\notin E(T)}}\exp(-\kappa_{i_m,i_n}/N)=1, \tag{5.14}$$

$$\lim_{N\to\infty}\binom{N-1}{\ell-1}\prod_{(m,n)\in E(T)}(1-\exp(-\kappa_{i_m,i_n}/N))=\frac{1}{(\ell-1)!}\prod_{(m,n)\in E(T)}\kappa_{i_m,i_n}. \tag{5.15}$$

Now to treat the expectation term in (5.13), we rewrite $\bar{p}_j^{N,\ell}$ as $\sum_{j=\ell+1}^N \mathbb{1}_{\{X_n=j\}}$, and recall that (X_n) defined at (5.11) are IID.

$$\begin{aligned}
\mathbb{E}\left[\prod_{m=1}^{\ell}\prod_{j=1}^{k}\exp\left(-\frac{\kappa_{i_m,j}\bar{p}_j^{N,\ell}}{N}\right)\right] &= \mathbb{E}\left[\prod_{m=1}^{\ell}\prod_{j=1}^{k}\prod_{n=\ell+1}^{N}\exp\Big(-\frac{\kappa_{i_m,j}\mathbb{1}_{\{X_n=j\}}}{N}\Big)\right]\\
&=\prod_{n=\ell+1}^{N}\mathbb{E}\left[\prod_{j=1}^{k}\prod_{m=1}^{\ell}\exp\Big(-\frac{\kappa_{i_m,j}\mathbb{1}_{\{X_n=j\}}}{N}\Big)\right]\\
&=\prod_{n=\ell+1}^{N}\left[1-\Phi+\sum_{j=1}^{k}\pi_j\prod_{m=1}^{\ell}\exp\Big(-\frac{\kappa_{i_m,j}}{N}\Big)\right]\\
&=\left[1-\Phi+\sum_{j=1}^{k}\pi_j\exp\Big(-\frac{\sum_{m=1}^{\ell}\kappa_{i_m,j}}{N}\Big)\right]^{N-\ell}\\
&=\left[1-\Phi+\sum_{j=1}^{k}\pi_j\Big(1-\frac{\sum_{m=1}^{\ell}\kappa_{i_m,j}}{N}+O(N^{-2})\Big)\right]^{N-\ell}\\
&=\left[1-\frac{\sum_{m=1}^{\ell}\Delta_{i_m}}{N}+O(N^{-2})\right]^{N-\ell}.
\end{aligned}$$

Recalling that ℓ is fixed, we obtain the limit

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{m=1}^{\ell}\prod_{j=1}^{k}\exp\left(-\frac{\kappa_{i_m,j}\bar{p}_j^{N,\ell}}{N}\right)\right] = \prod_{m=1}^{\ell}\exp(-\Delta_{i_m}). \tag{5.16}$$

Finally, we treat the extra term in (5.13), namely the probability that $C(1)$ includes a cycle. If $C(1)$ includes a cycle and $|C(1)| = \ell$, then it includes at least ℓ edges. So we can bound this probability as

$$\mathbb{P}\Big(|C(1)| = \ell \text{ and } C(1) \text{ includes a cycle in } \tilde{G}^N(\pi,\kappa)\Big) \le \tbinom{N-1}{\ell-1}\sum_{E=\ell}^{\binom{\ell}{2}}\tbinom{\binom{\ell}{2}}{E}\left(1-\exp\left(-\tfrac{\kappa_{\max}}{N}\right)\right)^E.$$

Recall again that ℓ is fixed, so $\binom{N-1}{\ell-1} = \Theta(N^{\ell-1})$. Each summand has magnitude $\Theta(N^{-E})$, so

$$\lim_{N\to\infty}\mathbb{P}\Big(|C(1)| = \ell \text{ and } C(1) \text{ includes a cycle in } \tilde{G}^N(\pi,\kappa)\Big) = 0. \tag{5.17}$$

Combining (5.17) with (5.16), (5.14) and (5.15),

$$\lim_{N\to\infty}\mathbb{P}\Big(|C(1)| = \ell \text{ in } \tilde{G}^N(\pi,\kappa)\Big) = \frac{1}{(\ell-1)!}\sum_{T\in T_{[\ell]}}\sum_{\substack{i_1,\dots,i_\ell\\ \in[k]}}\left[\prod_{(m,n)\in E(T)}\kappa_{i_m,i_n}\right]$$
$$\prod_{m\in[\ell]}\pi_{i_m}\exp(-\Delta_{i_m}).$$
$$\lim_{N\to\infty}\mathbb{P}\Big(|C(1)| = \ell \text{ in } \tilde{G}^N(\pi,\kappa)\Big) = \frac{1}{\ell!}\sum_{T\in T^{\rho}_{[\ell]}}\sum_{\substack{i_1,\dots,i_\ell\\ \in[k]}}\left[\prod_{(m,n)\in E(T)}\kappa_{i_m,i_n}\right] \tag{5.18}$$
$$\prod_{m\in[\ell]}\pi_{i_m}\exp(-\Delta_{i_m}),$$

where the second equality holds by considering the natural 1-to-ℓ mapping from $T_{[\ell]}$ to $T^{\rho}_{[\ell]}$, under which the summands are preserved.

However, we can also treat the LHS of (5.18) using Theorem 9.1 of [15], a local limit result for IRGs, which we now state in the language of this thesis. First, for any graph G, let $\mathcal{N}_\ell(G)$ be the number of vertices in G which lie in components of size exactly ℓ.

Then Theorem 9.1 of [15] states that whenever $p^N/N \to \pi \in \Pi_{\leq 1}$,

$$\frac{1}{N}\mathcal{N}_\ell\Big(G^N(p^N,\kappa)\Big) \quad \stackrel{\mathbb{P}}{\to} \quad \mathbb{P}(|\Xi^{\pi,\kappa}|=\ell). \tag{5.19}$$

Although Section 9 of [15] specifically excludes random graphs on what these authors term *generalised vertex spaces*, of which $\tilde{G}^N(\pi,\kappa)$ is an example, this is not a major problem. In $\tilde{G}^N(\pi,\kappa)$, consider the sequence $\bar{p}^{N,0} := (\bar{p}_1^{N,0},\dots \bar{p}_k^{N,0})$ as defined in (5.12), which records the number of vertices of each type present in the graph. Conditional on $\bar{p}^{N,0}$, $\tilde{G}^N(\pi,\kappa)$ has the same distribution on the space of graphs with k types, up to random relabelling of the vertices, as $G^N(\bar{p}^{N,0},\kappa)$. However, $\bar{p}^{N,0}/N$ converges in probability to π as $N\to\infty$. Therefore, since by construction $\mathcal{N}_\ell(\tilde{G}^N(\pi,\kappa))/N \leq 1$ almost surely, we can lift (5.19) to obtain

$$\frac{1}{N}\mathcal{N}_\ell\Big(\tilde{G}^N(\pi,\kappa)\Big) \quad \stackrel{\mathbb{P}}{\to} \quad \mathbb{P}(|\Xi^{\pi,\kappa}|=\ell), \tag{5.20}$$

and indeed this convergence holds in expectation also. But the possible vertices $[N]$ of $\tilde{G}^N(\pi,\kappa)$ are exchangeable by construction, and so

$$\mathbb{E}\Big[\mathcal{N}_\ell\Big(\tilde{G}^N(\pi,\kappa)\Big)\Big] = N\mathbb{P}\Big(|C(1)|=\ell \text{ in } \tilde{G}^N(\pi,\kappa)\Big).$$

From this, we obtain

$$\lim_{N\to\infty}\mathbb{P}\Big(|C(1)|=\ell \text{ in } \tilde{G}^N(\pi,\kappa)\Big) = \mathbb{P}(|\Xi^{\pi,\kappa}|=\ell). \tag{5.21}$$

Then, by combining (5.13) and (5.21), the required result (5.10) follows immediately. □

Now we are in a position to show that $(v_\ell(t))$ constructed from $\Xi^{(\pi(t),\kappa(t))}$ indeed satisfies the Smoluchowski equations. Recall we use the shorthand $\Xi^{(t)} := \Xi^{(\pi(t),\kappa(t)}$ when the type flow is fixed.

Proposition 5.13. Given $(\pi(t))_{t\geq 0}$ a frozen percolation type flow with initial kernel κ, set $v_\ell(t) = \mathbb{P}\Big(|\Xi^{(t)}|=\ell\Big)$ as before. Then $(v(t))_{t\geq 0}$ satisfies the Smoluchowski equations

(1.4), with $T_g = t_c$. Furthermore, we have

$$\sum_{l=1}^{\infty} v_l(t) = \sum_{i=1}^{k} \pi_i(t), \tag{5.22}$$

and so it is consistent to call both of these quantities $\Phi(t)$.

Proof. We show (5.22) first. We know that $\rho(\kappa(t) \circ \pi(t)) \leq 1$, which is precisely the condition required to conclude $\mathbb{P}\Big(|\Xi^{(t)}| = \infty\Big) = 0$, as in Proposition 3.7. So

$$\begin{aligned}\sum_{\ell=1}^{\infty} v_\ell(t) &= 1 - \mathbb{P}\Big(\Xi^{(t)} = \varnothing\Big) - \mathbb{P}\Big(|\Xi^{(t)}| = \infty\Big) \\ &= 1 - \mathbb{P}\Big(\Xi^{(t)} = \varnothing\Big) = \sum_{i=1}^{k} \pi_i(t).\end{aligned}$$

Now we consider the derivatives of $v_\ell(t)$. We write $\mu(t)$ as a shorthand for $\mu(\kappa(t) \circ \pi(t))$. We also write $\Delta_i(t) := \sum_{j=1}^{k} \kappa_{i,j}(t)\pi_j(t)$. First we observe that, for $t < t_c$,

$$\frac{\mathrm{d}\Delta_i(t)}{\mathrm{d}t} = 1, \quad \forall i \in [k],$$

and for $t > t_c$,

$$\begin{aligned}\frac{\mathrm{d}\Delta_i(t)}{\mathrm{d}t} &\overset{(5.4)}{=} \sum_{j=1}^{k} \pi_j(t) - \phi(t) \sum_{j=1}^{k} \kappa_{i,j}(t)\mu_j(t) \\ &= \Phi(t) - \phi(t)\frac{\mu_i(t)}{\pi_i(t)}, \quad i \in [k],\end{aligned} \tag{5.23}$$

from the definition of $\mu(t)$, and where by Lemma 5.11, $\pi_i(t) > 0$.

Then, from Lemma 5.12, $v_\ell(t)$ is given by:

$$\ell! v_\ell(t) = \sum_{T \in T^{\rho}_{[\ell]}} \sum_{\substack{i_1,\ldots,i_\ell \\ \in [k]}} \left[\prod_{(m,n) \in E(T)} \kappa_{i_m,i_n}(t) \right] \prod_{m=1}^{\ell} \pi_{i_m}(t) \exp(-\Delta_{i_m}(t)).$$

We differentiate directly with the product rule, and use (5.4) and (5.23). For brevity, we set

$$A(t) := \sum_{m=1}^{\ell} \Delta_{i_m}(t).$$

Note that $A(t)$ is a function of $i_1, \ldots, i_\ell$, though for brevity this dependence is suppressed. Then, for $t > t_c$,

$$\begin{aligned}
\ell! \frac{\mathrm{d}}{\mathrm{d}t} v_\ell(t) = &\sum_{T \in T^\rho_{[\ell]}} \sum_{\substack{i_1,\ldots,i_\ell \\ \in [k]}} \exp(-A(t)) \left[\prod_{(m,n)\in E(T)} \kappa_{i_m,i_n}(t) \right] \left[\phi(t) \sum_{m'=1}^{\ell} \frac{\mu_{i_{m'}}(t)}{\pi_{i_{m'}}(t)} - \ell\Phi(t) \right] \left[\prod_{m=1}^{\ell} \pi_{i_m}(t) \right] \\
&- \phi(t) \sum_{T \in T^\rho_{[\ell]}} \sum_{\substack{i_1,\ldots,i_\ell \\ \in [k]}} \exp(-A(t)) \left[\prod_{(m,n)\in E(T)} \kappa_{i_m,i_n}(t) \right] \sum_{m=1}^{\ell} \mu_{i_m}(t) \prod_{\substack{m'=1 \\ m'\neq m}}^{\ell} \pi_{i_m}(t) \\
&+ \sum_{T \in T^\rho_{[\ell]}} \sum_{\substack{i_1,\ldots,i_\ell \\ \in [k]}} \exp(-A(t)) \left[\sum_{(m,n)\in E(T)} \prod_{\substack{(m',n')\in E(T) \\ (m',n')\neq(m,n)}} \kappa_{i_{m'},i_{n'}}(t) \right] \prod_{m=1}^{\ell} \pi_{i_m}(t).
\end{aligned} \tag{5.24}$$

The first line comes from differentiating $\exp(-A(t))$ using (5.23); the second line from differentiating $\pi_{i_m}(t)$ using (5.4); and the final line from $\prod_{(m,n)\in E(T)} \kappa_{i_m,i_n}(t)$ directly. In the first two lines, the terms involving $\phi(t)$ cancel, leaving $-\ell \cdot \ell! v_\ell(t)\Phi(t)$. This applies equally for $t < t_c$, for which $\Phi(t) \equiv 1$, and for $t = t_c$, as the left- and right-derivatives match. To deal with the third line, for any $t \geq 0$, given T and $(m,n) \in E(T)$, consider the pair of disjoint trees T^m, T^n formed by removing the edge (m,n) from T, where $m \in T^m$ and $n \in T^n$. Then the sum in the third line of (5.24) splits as a product across these two trees:

$$\begin{aligned}
&\sum_{T \in T^\rho_{[\ell]}} \sum_{(m,n)\in E(T)} \sum_{\substack{i_1,\ldots,i_\ell \\ \in [k]}} \left[\prod_{\substack{(m',n')\in E(T) \\ (m',n')\neq(m,n)}} \kappa_{i_{m'},i_{n'}}(t) \right] \prod_{m=1}^{\ell} \pi_{i_m}(t) \exp(-\Delta_{i_m}(t)) \\
&= \sum_{T \in T^\rho_{[\ell]}} \sum_{(m,n)\in E(T)} \left(\sum_{\substack{i_{m'}\in[k] \\ m'\in T^m}} \left[\prod_{(m',n')\in E(T^m)} \kappa_{i_{m'},i_{n'}}(t) \right] \left[\prod_{m'\in T^m} \pi_{i_{m'}}(t) \exp\Big(\Delta_{i_{m'}}(t)\Big) \right] \right) \\
&\quad \times \left(\sum_{\substack{i_{n'}\in[k] \\ n'\in T^n}} \left[\prod_{(m',n')\in E(T^n)} \kappa_{i_{m'},i_{n'}}(t) \right] \left[\prod_{n'\in T^n} \pi_{i_{n'}}(t) \exp\Big(-\Delta_{i_{n'}}(t)\Big) \right] \right).
\end{aligned} \tag{5.25}$$

Consider the set of rooted trees on $[\ell]$ with an identified edge

$$\mathbb{T}_{[\ell]} := \Big\{ (T, (m,n)) \,:\, T \in T^\rho_{[\ell]}, (m,n) \in E(T) \Big\}.$$

Recall a *rooted forest* is a disjoint union of rooted trees. Let $T^{(2)}_{[\ell]}$ be the set of rooted forests on $[\ell]$ with exactly two trees. Consider the map from $\mathbb{T}_{[\ell]}$ to $T^{(2)}_{[\ell]}$ given by removing the identified edge (m,n) from T, and rooting the two resulting trees at m and n. It is immediately clear that this map is ℓ-to-1, since the root of T plays no role in the map!

So in (5.25), we may replace the double sum

$$\sum_{T\in T^{\rho}_{[\ell]}}\ \sum_{(m,n)\in E(T)} \quad \text{with the sum} \quad \ell \sum_{\substack{T^1\sqcup T^2\\ \in T^{(2)}_{[\ell]}}}.$$

Then, by considering which elements of $[\ell]$ belong to each of the two trees, we can replace the latter sum with

$$\frac{\ell}{2}\sum_{r=1}^{\ell-1}\ \sum_{A\in\binom{[\ell]}{r}}\ \sum_{T^1\in T^{\rho}_A}\ \sum_{T^2\in T^{\rho}_{[\ell]\setminus A}},$$

where recall T^{ρ}_A is the set of rooted graphs labelled by A. Note that in this sum, the $\frac{1}{2}$ appears because the order of trees T^1, T^2 does not matter. So we rewrite (5.25) as

$$\frac{\ell}{2}\sum_{r=1}^{\ell-1}\ \sum_{A\in\binom{[\ell]}{r}}\left(\sum_{T^1\in T^{\rho}_A}\ \sum_{\substack{i_{m'}\in[k]\\ m'\in T^1}}\left[\prod_{(m',n')\in E(T^1)}\kappa_{i_{m'},i_{n'}}(t)\right]\left[\prod_{m'\in T^1}\pi_{i_{m'}}(t)\exp\Big(-\Delta_{i_{m'}}(t)\Big)\right]\right)$$

$$\times\left(\sum_{T^2\in T^{\rho}_{[\ell]\setminus A}}\ \sum_{\substack{i_{n'}\in[k]\\ n'\in T^2}}\left[\prod_{(m',n')\in E(T^2)}\kappa_{i_{m'},i_{n'}}(t)\right]\left[\prod_{n'\in T^2}\pi_{i_{n'}}\exp\Big(-\Delta_{i_{n'}}(t)\Big)\right]\right).$$

We now relabel the variables inside each large bracket, and move factorials around, to obtain

$$\frac{\ell}{2}\cdot\ell!\sum_{r=1}^{\ell-1}\left(\frac{1}{r!}\sum_{T^1\in T^{\rho}_{[r]}}\ \sum_{\substack{i_1,\dots,i_r\\ \in[k]}}\left[\prod_{(m,n)\in E(T)}\kappa_{i_m,i_n}(t)\right]\prod_{m=1}^{r}\pi_{i_m}(t)\exp(-\Delta_{i_m}(t))\right)$$

$$\times\left(\frac{1}{(\ell-r)!}\sum_{T^2\in\mathcal{T}^{\rho}_{[\ell-r]}}\sum_{\substack{i_1,\dots,i_{\ell-r}\\ \in[k]}}\left[\prod_{(m,n)\in E(T)}\kappa_{i_m,i_n}(t)\right]\prod_{m=1}^{\ell-r}\pi_{i_m}(t)\exp(-\Delta_{i_m}(t))\right),$$

which is equal to

$$\ell!\cdot\frac{\ell}{2}\sum_{r=1}^{\ell-1}v_r(t)v_{\ell-r}(t).$$

We have already seen that the first two lines of (5.24) are equal to $-\ell\cdot\ell! v_\ell(t)\Phi(t)$. Therefore, cancelling the $\ell!$ terms, we conclude from (5.24) that

$$\frac{\mathrm{d}}{\mathrm{d}t}v_\ell(t)=\frac{\ell}{2}\sum_{r=1}^{\ell-1}v_r(t)v_{\ell-r}(t)-\ell\Phi(t)v_\ell(t),$$

for all $t\geq 0$, which is, up to a change of notation, as required. □

5.2.2 FP type flows are unique

Now we can finish the proof of Proposition 5.10.

Suppose we have FP type flows $\pi(\cdot)$ and $\nu(\cdot)$, with sums $\Phi^\pi(\cdot),\Phi^\nu(\cdot)$ respectively, and $\pi(0)=\nu(0)$. So we may consider the associated solutions to the Smoluchowski equations given by Proposition 5.13, $(v^\pi(\cdot)),(v^\nu(\cdot))$. Crucially, $\pi(0)=\nu(0)$ implies $v^\pi(0)=v^\nu(0)$. Theorem 4.1 concerning uniqueness of solutions to Smoluchowski's equations then gives $v^\pi(t)=v^\nu(t)$ for all times $t\geq 0$. Furthermore, from (5.22), $\Phi^\pi(t)=\Phi^\nu(t)$ for all $t\geq 0$, and $t_c^\pi=t_c^\nu$, with $\phi^\pi(t)=\phi^\nu(t)$ for all $t\geq t_c^\pi$.

We may now use the classical technique for verifying uniqueness of solutions to ODEs, since we have shown in Lemma 3.15 that μ is locally Lipschitz. The flow $\pi(\cdot)$ satisfies the integral version of (5.4),

$$\pi(t)=\pi(t_c)-\int_{t_c}^{t}\mu(\kappa(s)\circ\pi(s))|\mathrm{d}\Phi^\pi(s)|,\quad t\geq t_c, \tag{5.26}$$

and similarly for $\nu(\cdot)$. So

$$\pi(t)-\nu(t)=\int_{t_c}^{t}[\mu(\kappa(s)\circ\nu(s))-\mu(\kappa(s)\circ\pi(s))]|\mathrm{d}\Phi(s)|,\quad t\geq t_c.$$

For a fixed time $T > t_c$, by Lemma 5.11, we can choose $\eta > 0$ such that $\pi_i(T), \nu_i(T) \geq \eta$ for all $i \in [k]$. Now set $\delta := t_c \wedge \kappa_{\min}$. From the assumptions we made about the initial conditions, $\kappa_{i,j}(t) \geq \delta > 0$ whenever $t \geq t_c$. So with constant $C(\delta\eta, \kappa_{\max} + T)$ given by (3.9), for $t \in [t_c, T]$,

$$\begin{aligned} ||\mu(\kappa(t) \circ \pi(t)) - \mu(\kappa(t) \circ \nu(t))||_1 &\leq C(\delta\eta, \kappa_{\max} + T) \max_{i,j \in [k]} [\kappa(t) \circ \pi(t) - \kappa(t) \circ \nu(t)]_{i,j} \\ &\leq C(\delta\eta, \kappa_{\max} + T) \cdot (\kappa_{\max} + T) ||\pi(t) - \nu(t)||_1. \end{aligned} \tag{5.27}$$

Therefore, for $t \in [t_c, T]$,

$$||\pi(t) - \nu(t)||_1 \leq C(\delta\eta, T + \kappa_{\max}) \int_{t_c}^{t} ||\pi(s) - \nu(s)||_1 \phi(s) \mathrm{d}s.$$

We have $\pi(t_c) = \nu(t_c)$, so applying Gronwall's Lemma gives $\pi(t) = \nu(t)$ for all $t \in [t_c, T]$. But T was arbitrary, and so in fact we may conclude $\pi(t) = \nu(t)$ for all $t \geq 0$. This completes the proof of Proposition 5.10.

5.3 Proof of Theorem 5.7

We will prove Theorem 5.7 by considering weak limits in $\mathbb{D}^k([0,T])$ of the sequence of processes $(\pi^N(\cdot))$. We will show that the sequence is tight and that any weak limit is a frozen percolation type flow.

Recall we are given a kernel κ and an initial distribution $\pi(0)$ satisfying the conditions of Theorem 5.5, and thus there is a unique frozen percolation type flow $(\pi(t))_{t\geq 0}$ with these initial conditions. We are also given a family of multitype frozen percolation processes $(\mathcal{G}^N(t))_{t\geq 0}$. Associated to these is a collection of processes $(\pi^N(t))_{t\geq 0}$ recording the proportion of alive vertices of each type, for which $\pi^N(0) \xrightarrow{d} \pi(0)$ as $N \to \infty$. We also have $(\Phi^N(t))_{t\geq 0}$ which records the total proportion of alive vertices in $\mathcal{G}^N(t)$.

Analogously, we may also define

$$v_\ell^N(t) := \frac{1}{N}\#\Big\{\text{alive vertices in } \mathcal{G}^N(t) \text{ with component size } \ell\Big\}.$$

Recall the definition of the multitype branching process $\Xi^{\pi(0),\kappa}$, and as in the previous section, set

$$v_\ell(0) := \mathbb{P}\Big(|\Xi^{\pi(0),\kappa}| = \ell\Big).$$

Since $\mathcal{G}^N(0) \stackrel{d}{=} G^N(p^N,\kappa)$, we may use Theorem 9.1 from [15] which asserts that $v_\ell^N(0) \stackrel{d}{\to} v_\ell(0)$, for each $\ell \geq 1$. Then, by Scheffé's Lemma, we obtain

$$v^N(0) \stackrel{d}{\to} v(0), \quad \text{in } \ell_1 \text{ as } N \to \infty.$$

This is the condition we require to use Theorem 4.2, since $(v_\ell^N(\cdot), \ell \geq 1)$ are exactly the sequence of component-size densities in a family of frozen percolation processes.

In particular, we will use the consequence that $\Phi^N \to \Phi$, uniformly in distribution on $[0,T]$. Recall that $\Phi(t) = 1$ for $t \in [0,t_c]$, and Φ is strictly decreasing and uniformly continuous on (t_c,∞). Furthermore, from Proposition 5.13, $\Phi(t) = ||\pi(t)||_1$.

Outline of argument

First we check that the sequence of processes $(\pi^N(\cdot))$ is tight in $\mathbb{D}^k([0,T])$, and that every component of any weak limit is bounded away from zero. We deduce from Theorem 4.2 that weak limits are continuous and after t_c are strictly decreasing and critical. We will argue that supercritical period give rise to jumps, and subcritical periods are locally constant in the weak limits, and neither of these behaviours is allowed.

Finally, we show that any weak limit satisfies the equation (5.4). Our argument will be that in the limit the majority of mass is lost as a result of freezing large components, and the proportion of types within in a large component is well-approximated by the appropriate left-eigenvector, precisely as shown in Theorem 3.20.

5.3.1 Tightness and simple properties of weak limits

Throughout this section, we assume $T > 0$, and that both the initial kernel κ and the initial distribution $\pi(0)$ are fixed.

Tightness and Theorem 5.5

Note that each π^N is cadlag, and non-increasing, and $\pi^N(0)$ lies in a compact set, since it satisfies $\sum_{i=1}^k \pi_i^N(0) = 1$. It follows that the set of possible trajectories of any $(\pi^N(t))_{t\in[0,T]}$ is compact in $\mathbb{D}^k([0,T])$, and so certainly the sequence of processes $(\pi^N(\cdot))$ is tight.

Therefore, $(\pi^N(\cdot))$ has weak limits. The remainder of this proof of Theorem 5.7 establishes that any such weak limit satisfies the conditions of Definition 5.4 to be a frozen percolation type flow with the correct initial conditions. As a result, the full statement of Theorem 5.5 follows from Section 5.2 and the proof of Theorem 5.7 to follow in this section.

Before t_c

From now on, let π be any weak limit of (π^N) in $\mathbb{D}^k([0,T])$ as $N \to \infty$. We will show that π is the (unique) frozen percolation type flow with initial kernel κ and initial distribution $\pi(0)$.

We know that $\Phi^N \to 1$ on $[0, t_c]$. Therefore, since for each $i \in [k]$, π_i^N is non-increasing, the same must be true for each π_i. Therefore $\sum_{i\in[k]} \pi_i^N(t) \to 1$ for $t \in [0, t_c]$ implies $\pi^N(t) \to \pi(t)$ for the same range of t. In particular, any weak limit π satisfies $\pi(t) = \pi(0)$ for $t \in [0, t_c]$, as required.

Continuity

Again, we know $\Phi^N \to \Phi$, which is continuous. Any weak limit π must satisfy $||\pi(t)||_1 = \Phi(t)$ for $t \in [0,T]$, and every component $\pi_i(t)$ is non-increasing with t. Therefore, if with positive probability, for some $i \in [k]$, $\pi_i(\cdot)$ has a (downward) jump, so does $\Phi(\cdot)$. This is a contradiction, and thus $\pi(\cdot)$ is almost surely continuous.

Lower bounds on $\pi^N(T)$

As in the analysis of type flows, in order to use the Lipschitz condition, it is convenient to show the following lemma, which asserts that the proportion of alive vertices of each type is bounded below in probability uniformly on compact time intervals.

Lemma 5.14. For any $T > 0$, there exists $\eta = \eta(T) > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}\Big(\exists i \in [k] \text{ s.t. } \pi_i^N(T) < \eta\Big) = 0. \tag{5.28}$$

Proof. We consider the proportion of isolated alive vertices of type i in the frozen percolation process, as a lower bound on the proportion of all alive vertices of type i. We use a second-moment method, under a coupling with the classical Erdős–Rényi dynamics with no freezing.

Each possible edge carries an exponential clock with parameter $1/N$. Because of the dynamics of the frozen percolation process, sometimes we do not add the edge when the corresponding clock rings, because at least one of the incident vertices is already frozen. We say a vertex v is *highly isolated* at time T if it was isolated in $\mathcal{G}^N(0)$, and none of the $N-1$ clocks on edges incident to v ring before time T. Certainly if a vertex is highly isolated, then it is also isolated, provided it is alive.

Let v be a uniformly-chosen vertex in $[N]$, and let $\mathcal{H}_v^N(T,i)$ be the event that v has type i, and is alive and highly isolated at time T in $\mathcal{G}^N(T)$. For $\mathcal{H}_v^N(T,i)$ to hold, v must be assigned type i; and v must be isolated in the initial graph $\mathcal{G}^N(0)$; and none of the $N-1$ clocks on edges incident to v may ring before time T; and v must not be struck by lightning. So

$$\mathbb{P}\Big(\mathcal{H}_v^N(T,i) \,|\, \pi^N(0)\Big) = \pi_i^N(0)\left(\prod_{j=1}^k \exp\Big(-\kappa_{i,j}\Big[\pi_j^N(0) - \tfrac{1}{N}\mathbb{1}_{\{i=j\}}\Big]\Big)\right)\cdot\Big(e^{-T/N}\Big)^{N-1}\cdot e^{-\lambda(N)T},$$

and since $\pi^N(0) \xrightarrow{d} \pi(0)$ as $N \to \infty$, we have

$$\mathbb{P}\Big(\mathcal{H}_v^N(T,i)\Big) \to \pi_i(0)\alpha_i, \qquad \text{where } \alpha_i := \exp\left(-T - \sum_{j=1}^k [\kappa \circ \pi(0)]_{i,j}\right).$$

Now let v, w be a uniformly chosen pair of distinct vertices in $[N]$, and let $\mathcal{H}_{v,w}^N(T,i)$ be the event that both v and w have type i, and are alive and highly isolated at time T. There are $2N-3$ edges in $[N]^{(2)}$ incident to at least one of v and w. Also, conditional on the initial type distribution, and the event that v and w both have type i, there are $1 + 2[N\pi_i^N(0) - 2]$ possible edges between one of v and w and some other vertex with type i. (Note that we avoid double-counting edge vw.) So,

$$\mathbb{P}\Big(\mathcal{H}_{v,w}^N(T,i) \,|\, \pi^N(0)\Big) = \pi_i^N(0)\Big[\pi_i^N(0) - \tfrac{1}{N}\Big]\left(\prod_{j=1}^k \exp\Big(-\kappa_{i,j}\Big[2\pi_j^N(0) - \tfrac{3}{N}\mathbb{1}_{\{i=j\}}\Big]\Big)\right) \\ \times \Big(e^{-T/N}\Big)^{2N-3} \cdot e^{-2\lambda(N)T},$$

from which as before we have, as $N \to \infty$,

$$\mathbb{P}\Big(\mathcal{H}_{v,w}^N(T,i)\Big) \to \pi_i(0)^2\alpha_i^2.$$

Now let $H^N(T,i)$ be the number of alive, highly isolated vertices with type i in $\mathcal{G}^N(T)$. We have $\mathbb{E}\Big[\frac{H^N(T,i)}{N}\Big] \to \pi_i(0)\alpha_i$ and $\operatorname{var}\Big(\frac{H^N(T,i)}{N}\Big) \to 0$.

So for any $\eta \in (0, \pi_i(0)\alpha_i)$, applying Chebyshev's inequality to $\frac{H^N(T,i)}{N}$,

$$\limsup_{N\to\infty} \mathbb{P}\Big(\pi_i^N(T) < \eta\Big) \le \limsup_{N\to\infty} \mathbb{P}\Big(\tfrac{H^N(T,i)}{N} < \eta\Big) \le \limsup_{N\to\infty} \frac{\operatorname{var}\Big(\frac{H^N(T,i)}{N}\Big)}{(\pi_i(0)\alpha_i - \eta)^2} = 0.$$

The statement (5.28) follows by taking $\eta < \min_i \pi_i(0)\alpha_i$. □

5.3.2 Weak limits are critical after t_c

We now show that for any weak limit π, the criticality condition $\rho(\kappa(t) \circ \pi(t)) = 1$ holds for all $t \ge t_c$. We first show that this eigenvalue cannot ever be greater than one, and then that it cannot be less than one. In both cases, the argument is by contradiction. If $\mathcal{G}^N(t)$ is ever supercritical, then with high probability giant components will be frozen, and so weak limits of Φ^N will not be continuous. If $\mathcal{G}^N(t)$ is subcritical, then not enough vertices will be frozen to ensure weak limits of Φ^N are strictly decreasing.

Weak limits are never supercritical

Proposition 5.15. For any $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} \rho(\kappa(t) \circ \pi(t)) \geq 1 + \epsilon \right) = 0. \tag{5.29}$$

Proof. The principal eigenvalue $\rho(\cdot)$ is continuous. The kernel $\kappa(\cdot)$ is continuous, and we have shown that $\pi(\cdot)$ is almost surely continuous. On the event $\{\sup_{t\in[0,T]} \rho(\pi(t)\circ\kappa(t)) \geq 1 + \epsilon\}$, either π has a discontinuity, or there exists a time-interval of positive width, during which $\rho \geq 1 + \epsilon/2$. So either (5.29) holds, or there exists a fixed time $s \in [0, T)$, and an infinite subsequence $\mathcal{N} \subseteq \mathbb{N}$ such that

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \mathbb{P}\Big(\rho\Big(\kappa(s) \circ \pi^N(s)\Big) \geq 1 + \epsilon/2\Big) > 0. \tag{5.30}$$

We assume (5.30) holds, and apply Lemma 3.12. We obtain that there exist $M \in \mathbb{N}$ and $\pi^{(1)}, \ldots, \pi^{(M)} \in \Pi_{\leq 1}$ and kernels $\kappa^{(1)}, \ldots, \kappa^{(m)} \in \mathbb{R}^{k\times k}_{\geq 0}$ such that $\rho(\kappa^{(m)}\circ\pi^{(m)}) = 1+\epsilon/3$, and for any $\pi \in \Pi_{\leq 1}$ and $\kappa \in [0, \kappa_{\max} + T]^{k\times k}$ with $\rho(\kappa \circ \pi) \geq 1 + \epsilon/2$, there exists $m \in [M]$ such that $\pi^{(m)} \leq \pi$ and $\kappa^{(m)} \leq \kappa$.

Recall that $(\mathcal{F}^N(t))_{t\geq 0}$ is the natural filtration of $(\pi^N(t))$. In particular, the event $\{\rho(\kappa(s) \circ \pi^N(s)) \geq 1 + \epsilon/2\}$ is $\mathcal{F}^N(s)$-measurable. On this event, at least one of the events $\{\pi^{(m)} \leq \pi^N(s)\}$ holds. From Proposition 5.2, conditional on $\mathcal{F}^N(s)$, up to labelling, $\mathcal{G}^N(s)$ has the same distribution as $G^N(N\pi^N(s), \kappa)$. Therefore, for any $\theta \in (0, 1)$,

$$\begin{aligned} &\mathbb{P}\Big(L_1\Big(\mathcal{G}^N(s)\Big) \geq \theta N \,\Big|\, \rho(\kappa(s) \circ \pi^N(s)) \geq 1 + \epsilon/2\Big) \\ &\qquad \geq \min_{m\in[M]} \mathbb{P}\Big(L_1\Big(G^N(\lfloor N\pi^{(m)}\rfloor, \kappa^{(m)})\Big) \geq \theta N\Big), \end{aligned} \tag{5.31}$$

where the floor function is applied component-wise. However, for each $m \in [M]$, Theorem 3.8 controls the asymptotic size of the largest component in the family of graphs on the RHS. That is, as $N \to \infty$,

$$\frac{1}{N} L_1\Big(G^N((\lfloor N\pi^{(m)} \rfloor, \kappa^{(m)})\Big) \quad \xrightarrow{d} \quad \mathbb{P}\Big(|\Xi^{\pi^{(m)},\kappa^{(m)}}| = \infty\Big),$$

for each $m \in [M]$, and furthermore this limit is equal to $\sum_{i\in[k]} \pi_i \zeta_i^{\pi^{(m)},\kappa^{(m)}}$, which is positive. We take $\theta > 0$ satisfying

$$\theta < \min_{m\in[M]} \mathbb{P}\Big(|\Xi^{\pi^{(m)},\kappa^{(m)}}| = \infty\Big).$$

Returning to (5.31) with this value of θ, we find

$$\lim_{N\to\infty} \mathbb{P}\Big(L_1\Big(\mathcal{G}^N(s)\Big) \geq \theta N \,\Big|\, \rho(\kappa(s) \circ \pi^N(s)) \geq 1 + \epsilon/2\Big) = 1.$$

So, if (5.30) holds, we have

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \mathbb{P}\Big(L_1\Big(\mathcal{G}^N(s)\Big) \geq \theta N\Big) > 0.$$

Conditional on the event $\Big\{L_1\Big(\mathcal{G}^N(s)\Big) \geq \theta N\Big\}$, the probability that this largest component is not struck by lightning before any fixed time $s' > s$ vanishes as $N \to \infty$, since the lightning rate $\lambda(N) \gg \frac{1}{N}$. So

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \mathbb{P}\Big(\Phi^N(s+) \leq \Phi^N(s) - \theta\Big) > 0,$$

and so the same holds for the limit. That is,

$$\mathbb{P}(\Phi(s+) \leq \Phi(s) - \theta) > 0,$$

which contradicts the almost sure continuity of any weak limit Φ. (5.29) then follows by contradiction. □

Weak limits are not subcritical after t_c

We start with a lemma concerning the expected size of the component of a uniformly-chosen vertex in a subcritical inhomogeneous random graph. The final step includes a bound which is rather weak, but will be sufficient for the main proposition which follows.

Lemma 5.16. Fix $N \in \mathbb{N}$, $p \in \mathbb{N}_0^k$ and $\kappa \in \mathbb{R}_{\geq 0}^{k\times k}$ satisfying $\rho(\kappa \circ p/N) < 1$. Let $C(v)$ be the component containing a uniformly chosen vertex in $G^N(p,\kappa)$. Then

$$\mathbb{E}[|C(v)|] \leq \frac{1}{\kappa_{\min}||\pi||_1} \cdot \frac{\rho(\kappa\circ\pi)}{1-\rho(\kappa\circ\pi)},$$

where $\kappa_{\min} := \min_{i,j\in[k]} \kappa_{i,j}$, and $\pi := p/N$.

Proof. Set $\bar{\pi} := \pi/||\pi||_1 = p/||p||_1$. Recall $\bar{\Xi}^{\pi,\kappa}$, from Definition 3.17, the multitype branching process where the root exists with probability one, and has type distribution given by $\bar{\pi}$. From Proposition 3.18, $\mathbb{E}[|C(v)|] \leq \mathbb{E}\Big[|\bar{\Xi}^{p/N,\kappa}|\Big] = \mathbb{E}\big[|\Xi^{\bar{\pi},\kappa}|\big]$. Now consider the multitype branching process tree $\hat{\Xi}^{\bar{\pi},\kappa}$, where the type of the root is given instead by the distribution $\mu(\kappa\circ\pi)$, and the offspring distributions are the same as for $\Xi^{\pi,\kappa}$ and $\bar{\Xi}^{\pi,\kappa}$. By considering the number of offspring at each generation of $\hat{\Xi}^{\pi,\kappa}$ we have

$$\mathbb{E}\Big[|\hat{\Xi}^{\pi,\kappa}|\Big] = 1 + \rho(\kappa\circ\pi) + \rho(\kappa\circ\pi)^2 + \ldots = \frac{1}{1-\rho(\kappa\circ\pi)}.$$

However, the distribution of $\bar{\Xi}^{\pi,\kappa}$ conditional on the root having type i is the same as the distribution of $\hat{\Xi}^{\pi,\kappa}$ conditional on the root having type i. Therefore, by the law of total expectation,

$$\mathbb{E}\Big[|\bar{\Xi}^{\pi,\kappa}|\Big] \leq \max_{i\in[k]} \frac{\pi_i/||\pi||_1}{\mu_i(\kappa\circ\pi)} \mathbb{E}\Big[|\hat{\Xi}^{\pi,\kappa}|\Big].$$

However, since $\mu(\kappa\circ\pi)$ is a left-eigenvector, we have

$$\frac{\mu_j(\kappa\circ\pi)}{\pi_j} = \frac{1}{\rho(\kappa\circ\pi)} \sum_{i=1}^k \mu_i(\kappa\circ\pi)\kappa_{i,j} \geq \frac{\kappa_{\min}}{\rho(\kappa\circ\pi)}, \quad j\in[k],$$

and the result follows immediately. ☐

Proposition 5.17. For any $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in[t_c,T]} \rho(\pi(t)\circ\kappa(t)) \le 1-\epsilon \right) = 0. \tag{5.32}$$

Proof. By the same argument as in Proposition 5.15, either (5.32) holds, or there exists $s \in [t_c, T)$ and an infinite subsequence $\mathcal{N} \subseteq \mathbb{N}$ such that

$$\liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \mathbb{P}\Big(\rho\Big(\kappa(s)\circ\pi^N(s)\Big) \le 1-\epsilon/2\Big) > 0.$$

Now, we can choose $\delta > 0$ such that $s+\delta < T$ and

$$\frac{\kappa_{i,j}(s+\delta)}{\kappa_{i,j}(s)} \le \frac{1-\epsilon/3}{1-\epsilon/2}, \quad \forall i,j \in [k].$$

Since $\rho(\cdot)$ is increasing as a function of each entry of its argument (Corollary 3.11),

$$\liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \mathbb{P}\Big(\rho\Big(\kappa(s+\delta)\circ\pi^N(s)\Big) \le 1-\epsilon/3\Big) > 0. \tag{5.33}$$

We now consider how many vertices are frozen during the time-interval $[s, s+\delta]$, in expectation. By construction of the lightning processes, and Proposition 5.2, for any $t > 0$, it is the case that conditional on $\mathcal{F}^N(t-)$ and the event that an alive vertex is struck by lightning at time t, the number of vertices frozen $\Big[\Phi^N(t-) - \Phi^N(t)\Big]$ has the same law as $|C(1)|$ in the IRG $G^N(N\pi^N(t-), \kappa(t))$. In particular, in our setting, for any lightning strike on an alive vertex at time $s' \in [s, s+\delta]$,

$$\begin{aligned}
&\mathbb{E}\Big[\Phi^N(s'-) - \Phi^N(s') \,\Big|\, \mathcal{F}^N(s'-), \Phi^N(s'-) - \Phi^N(s') > 0\Big] \\
&\quad\le \frac{1}{N}\mathbb{E}\Big[|C(1)| \text{ in } G^N(N\pi^N(s'-), \kappa(s')) \,\Big|\, \mathcal{F}^N(s'-)\Big] \\
&\quad\le \frac{1}{N}\mathbb{E}\Big[|C(1)| \text{ in } G^N(N\pi^N(s), \kappa(s+\delta)) \,\Big|\, \mathcal{F}^N(s'-)\Big],
\end{aligned}$$

almost surely, since $|C(1)|$ is an increasing function of the graphs. But the quantity in the final expectation is actually $\mathcal{F}^N(s)$-measurable, and so

$$\mathbb{E}\Big[\Phi^N(s'-) - \Phi^N(s') \,\Big|\, \mathcal{F}^N(s'-), \Phi^N(s'-) - \Phi^N(s') > 0\Big]$$

$$\le \frac{1}{N}\mathbb{E}\Big[|C(1)| \text{ in } G^N(N\pi^N(s), \kappa(s+\delta)) \,\Big|\, \mathcal{F}^N(s)\Big].$$

In particular, this upper bound is independent of behaviour on the interval $[s, s')$. The process recording all lightning strikes on alive vertices is dominated by a Poisson process with rate $N\lambda(N)$, so we obtain

$$\mathbb{E}\Big[\Phi^N(s) - \Phi^N(s+\delta) \,\Big|\, \mathcal{F}^N(s)\Big] \le \delta\lambda(N)\mathbb{E}\Big[|C(1)| \text{ in } G^N(N\pi^N(s), \kappa(s+\delta)) \,\Big|\, \mathcal{F}^N(s)\Big]. \tag{5.34}$$

Using Lemma 5.16, the expectation of this component size conditional on $\mathcal{F}^N(s)$ is at most

$$\frac{1}{\delta||\pi^N(s)||_1} \cdot \frac{\rho\Big(\kappa(s+\delta) \circ \pi^N(s)\Big)}{1 - \rho(\kappa(s+\delta) \circ \pi^N(s))},$$

almost surely. Consider η as given by Lemma 5.14, and define the event

$$\mathcal{A}^N := \Big\{\rho\Big(\kappa(s+\delta) \circ \pi^N(s)\Big) \le 1 - \epsilon/3,\ \Phi^N(s) \ge k\eta\Big\},$$

which is certainly $\mathcal{F}^N(s)$-measurable. By (5.28) and (5.33), as $\mathcal{N} \ni N \to \infty$, $\liminf \mathbb{P}\Big(\mathcal{A}^N\Big) > 0$. But then using (5.34) we have

$$\mathbb{E}\Big[\Phi^N(s) - \Phi^N(s+\delta) \,\Big|\, \mathcal{A}^N\Big] \le \frac{1}{k\delta\eta} \cdot \frac{3}{\epsilon} \cdot \delta\lambda(N) \ll 1.$$

It follows by the law of total probability and by Markov's inequality that for any $\theta > 0$,

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \mathbb{P}\Big(\Phi^N(s) - \Phi^N(s+\delta) \le \theta\Big) > 0,$$

and so

$$\mathbb{P}(\Phi(s) - \Phi(s+\delta) = 0) > 0,$$

which contradicts the requirement that any weak limit Φ is almost surely strictly decreasing on $[t_c, T]$. □

We have shown that any weak limit π is continuous, and satisfies $\rho(\kappa(t) \circ \pi(t)) = 1$ for $t \geq t_c$, and satisfies $\pi(t) = \pi(0)$ for $t \leq t_c$. Thus we have shown that (5.2) and (5.3) hold.

5.3.3 Asymptotic proportions of types of frozen vertices

To complete the proof of Theorem 5.7 it remains to show that any weak limit satisfies (5.4).

Weak convergence towards integral equation

Throughout this final section, κ is fixed, and so $\kappa(t)$ is fixed for all $t \geq 0$. To emphasise this, and for brevity, we will write $\mu(t, \pi(t))$ for $\mu(\kappa(t) \circ \pi(t))$ here.

Suppose we have

$$\sup_{t \in [t_c, T]} \left|\left| \pi^N(t_c) - \pi^N(t) + \int_{t_c}^{t} \mu(s, \pi^N(s-)) \mathrm{d}\Phi^N(s) \right|\right|_1 \xrightarrow{\mathbb{P}} 0, \tag{5.35}$$

as $N \to \infty$. We will show that this is a sufficient requirement for any weak limit $\pi(\cdot)$ to satisfy the following integral version of the differential equation (5.4) governing the evolution of the type distribution:

$$\pi(t_c) - \pi(t) + \int_{t_c}^{t} \mu(s, \pi(s)) \mathrm{d}\Phi(s) = 0, \quad t \in [t_c, T]. \tag{5.36}$$

This is sufficient for (5.4) since Φ is differentiable on (t_c, ∞), and $\mu(s, \pi(s))$ is almost surely continuous. We have $\Phi^N \to \Phi$ uniformly on $[0, T]$, and again let $\pi(\cdot)$ be a weak limit of $\pi^N(\cdot)$ along the subsequence $\mathcal{N} \subseteq \mathbb{N}$. Since $\pi(\cdot)$ and $\kappa(\cdot)$ are continuous, $\mu(\cdot, \pi(\cdot))$ is uniformly continuous on $[0, T]$. Therefore

$$\sup_{t \in [t_c, T]} \left|\left| \int_{t_c}^{t} \mu(s, \pi(s)) \mathrm{d}\Big[\Phi^N(s) - \Phi(s)\Big] \right|\right|_1 \xrightarrow{\mathbb{P}} 0,$$

as $N \to \infty$. To conclude (5.36) from (5.35), it remains to show that

$$\sup_{t\in[t_c,T]} \left|\left| \int_{t_c}^{T} \Big[\mu(s,\pi^N(s-)) - \mu(s,\pi(s))\Big] \mathrm{d}\Phi^N(s) \right|\right|_1 \xrightarrow{\mathbb{P}} 0, \tag{5.37}$$

as $\mathcal{N} \ni N \to \infty$. But certainly for any $t \in [t_c, T]$ we have

$$\left|\left| \int_{t_c}^{t} \Big[\mu(s,\pi^N(s-)) - \mu(s,\pi(s))\Big] \mathrm{d}\Phi^N(s) \right|\right|_1 \le \int_{t_c}^{T} \Big|\Big|\mu(s,\pi^N(s-)) - \mu(s,\pi(s))\Big|\Big|_1 \mathrm{d}\Phi^N(s).$$

Consider $\eta > 0$ as given by Lemma 5.14. It follows directly from (5.28) that

$$\mathbb{P}(\exists i \in [k] \text{ s.t. } \pi_i(T) < \eta) = 0.$$

Conditional on $\pi_i^N(s-) \ge \eta$ for all $i \in [k]$, Lemma 3.15 gives, as in (5.27),

$$\Big|\Big|\mu(s,\pi^N(s-)) - \mu(s,\pi(s))\Big|\Big|_1 \le (\kappa_{\max} + T) C(\eta(t_c \vee \kappa_{\min}), \kappa_{\max} + T) ||\pi^N(s-) - \pi(s)||_1.$$

Therefore, writing C for $(\kappa_{\max} + T)C(\eta(t_c \vee \kappa_{\min}), \kappa_{\max} + T)$, on the event $\{\pi_i^N(T) \ge \eta, \forall i \in [k]\}$,

$$\left|\left| \int_{t_c}^{t} \Big[\mu(s,\pi^N(s-)) - \mu(s,\pi(s))\Big] \mathrm{d}\Phi^N(s) \right|\right|_1 \le C \int_{t_c}^{T} ||\pi^N(s-) - \pi(s)||_1 \mathrm{d}\Phi^N(s).$$

As $\mathcal{N} \ni N \to \infty$, both $\pi^N \to \pi$, and $\Phi^N \to \Phi$ uniformly in distribution on $[0, T]$, so the RHS vanishes in probability. By Lemma 5.14, $\mathbb{P}\Big(\pi_i^N(T) \ge \eta \; \forall i \in [k]\Big) \to 1$. Thus (5.37) follows, and we may conclude (5.36) from (5.35). It remains to show (5.35). We will show (5.35) in the next section, after a preliminary result.

A result about coupled processes

First, we show a result about coupled processes which we will use to finish this proof.

The motivation for the setup is the following. Every time a component is frozen in the multitype frozen percolation process, the distribution of types in this frozen component is not *exactly* the same as the left-eigenvector of the appropriate kernel, but the difference is close to zero so long as the component is fairly large. The expression on the LHS of

(5.35) records the accumulation of this error. Each time Φ^N has a downward jump, the expected extra error accumulated is small relative to the expected size of the jump of Φ^N. The following result will show that this is enough to conclude that the total error is small in probability, uniformly in time.

For some $N \in \mathbb{N}$, consider $(\xi_m)_{0\leq m\leq N}$ and $(Y_m)_{0\leq m\leq N}$, $\mathbb{R}$-valued processes adapted to a filtration $\mathcal{F} = (\mathcal{F}_m)_{0\leq m\leq N}$. We will assume that $\xi_0 = Y_0 = 0$, and that (ξ_m) is non-decreasing. We will assume also that $\xi_N \leq 1$, and that for some $\delta \in (0,1)$ and $K \in \mathbb{N}$,

$$|Y_{m+1} - Y_m| \leq \xi_{m+1} - \xi_m \leq \frac{1}{K^2}, \quad \text{a.s.} \quad m = 0, 1, \ldots, N-1, \tag{5.38}$$

$$\text{and} \quad \big|\mathbb{E}[Y_{m+1} - Y_m|\mathcal{F}_m]\big| \leq \delta\mathbb{E}[\xi_{m+1} - \xi_m|\mathcal{F}_m], \quad \text{a.s.} \quad m = 0, 1, \ldots, N-1. \tag{5.39}$$

That is, the increments of ξ are bounded, and dominate the increments of Y. Furthermore the increments of Y have smaller expectation than those of ξ, uniformly in time and the history of the process.

Lemma 5.18. Whenever (5.38) and (5.39) hold, we have:

$$\mathbb{E}\left[\sup_{0\leq m\leq N} |Y_n|\right] \leq \frac{2}{K} + \delta. \tag{5.40}$$

Proof. We consider the Doob–Meyer decomposition of the process (Y_m). That is,

$$W_0 := 0, \qquad W_{m+1} := W_m + Y_{m+1} - \mathbb{E}\big[Y_{m+1}\,\big|\,\mathcal{F}_m\big], \quad m \geq 0,$$

$$A_0 := 0, \qquad A_{m+1} := A_m + \mathbb{E}\big[Y_{m+1} - Y_m\,\big|\,\mathcal{F}_m\big], \quad m \geq 0,$$

for which (W_m) is an $\mathcal{F}$-martingale, and (A_m) is a predictable process, and $Y_m = W_m + A_m$. All the statements which follow hold almost surely. First we consider (A_m). Using (5.39), we have

$$|A_{m+1} - A_m| \leq \delta\mathbb{E}\big[\xi_{m+1} - \xi_m\,\big|\,\mathcal{F}_m\big],$$

from which,

$$\mathbb{E}\left[\sup_{0\le m\le N}|A_m|\right] \le \mathbb{E}\left[\sum_{m=0}^{N-1}|A_{m+1}-A_m|\right] \le \delta\mathbb{E}[\xi_N] \le \delta. \tag{5.41}$$

Now we turn to (W_m). Certainly, for any $0 \le m \le N-1$, conditional on $\mathcal{F}_m$,

$$W_{m+1} - W_m = Y_{m+1} - Y_m - \mathbb{E}\big[Y_{m+1} - Y_m \,\big|\, \mathcal{F}_m\big],$$

and so

$$\mathbb{E}\Big[(W_{m+1}-W_m)^2 \,\Big|\, \mathcal{F}_m\Big] \le \mathbb{E}\Big[(Y_{m+1}-Y_m)^2 \,\Big|\, \mathcal{F}_m\Big].$$

Using (5.38), for any $0 \le m \le N-1$,

$$\mathbb{E}\Big[(W_{m+1}-W_m)^2 \,\Big|\, \mathcal{F}_m\Big] \le \mathbb{E}\Big[(\xi_{m+1}-\xi_m)^2 \,\Big|\, \mathcal{F}_m\Big] \le \frac{1}{K^2}\mathbb{E}\big[\xi_{m+1}-\xi_m \,\big|\, \mathcal{F}_m\big].$$

Then, since (W_m) is a martingale bounded in L^2, by orthogonality of increments (see §12.1 of [71]),

$$\mathbb{E}\Big[W_N^2\Big] = \sum_{m=0}^{N-1}\mathbb{E}\Big[(W_{m+1}-W_m)^2\Big] \le \frac{1}{K^2}\sum_{m=0}^{N-1}\mathbb{E}[\xi_{m+1}-\xi_m] \le \frac{1}{K^2},$$

since $\xi_N \le 1$. Finally, using Doob's L^2 inequality,

$$\mathbb{E}\left[\sup_{0\le m\le N}|W_m|\right] \le \sqrt{\mathbb{E}\left[\sup_{0\le m\le N} W_m^2\right]} \le \sqrt{4\mathbb{E}\big[W_N^2\big]} \le \frac{2}{K}. \tag{5.42}$$

Since $Y_m = W_m + A_m$, it follows immediately from (5.41) and (5.42) that

$$\mathbb{E}\left[\sup_{0\le m\le N}|Y_m|\right] \le \frac{2}{K} + \delta,$$

as required. □

5.3.4 Decomposition via freezing times

We now prove (5.35), which is equivalent to

$$\sup_{t\in[t_c,T]}\left|\left|\int_{t_c}^{t}\Big[\mu(s,\pi^N(s-))\mathrm{d}\Phi^N(s)-\mathrm{d}\pi^N(s)\Big]\right|\right|_1 \xrightarrow{\mathbb{P}} 0. \tag{5.43}$$

To address this, we categorise each frozen vertex by its type and by its distance from the associated vertex which was struck by lightning. This will allow us to use the results shown in Section 3.3. In the process $\mathcal{G}^N$, for each vertex $v\in[N]$, say s_v is the time at which v is frozen, as a result of some vertex w being struck by lightning. (Note that w is possibly v itself.) Define $d(v)=d(v,w)$ to be the graph distance in $\mathcal{G}^N(s_v)$ between v and w.

We now define for each $i\in[k]$ and any $r\in\{0,1,\dots,N-1\}$,

$$\Psi^N(r,i,t)=\frac{1}{N}\#\{v\in[N]:\mathrm{type}(v)=i,\ s_v\in[0,t] \text{ and } d(v)=r\}. \tag{5.44}$$

Also define $\Psi^N(r,t):=\sum_{i=1}^{k}\Psi^N(r,i,t)$, that is, the total proportion of vertices of any type frozen up to time t which were distance r from the vertex struck by lightning. We have

$$\sum_{r=0}^{N-1}\mathrm{d}\Psi^N(r,s)=-\mathrm{d}\Phi^N(s),\qquad \sum_{r=0}^{N-1}\mathrm{d}\Psi^N(r,i,s)=-\mathrm{d}\pi_i^N(s),$$

and so (5.43) is further equivalent to

$$\sup_{t\in[t_c,T]}\left|\int_{t_c}^{t}\sum_{r=0}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s)-\mathrm{d}\Psi^N(r,i,s)\Big]\right| \xrightarrow{\mathbb{P}} 0,\quad \forall i\in[k],$$

as $N\to\infty$. Therefore, to prove (5.35) and complete the proof of Theorem 5.7 it will suffice to show the following lemma.

Lemma 5.19. For each type $i\in[k]$,

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in[t_c,T]}\left|\int_{t_c}^{t}\sum_{r=0}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s)-\mathrm{d}\Psi^N(r,i,s)\Big]\right|\right]=0. \tag{5.45}$$

Proof. Throughout the proof, we fix $i \in [k]$. We start by showing that small values of r do not contribute on this scale in the limit. Fix some $R \in \mathbb{N}$. Then, at any time $t \le T$, the expected number of vertices within distance $R-1$ of a uniformly chosen alive vertex in $\mathcal{G}^N(t)$ is at most

$$1 + \Big[N(1 - e^{-(\kappa_{\max}+T)/N})\Big] + \ldots + \Big[N(1 - e^{-(\kappa_{\max}+T)/N})\Big]^{R-1}.$$

Therefore

$$\mathbb{E}\left[\sum_{r=0}^{R-1} \Psi^N(r,T)\right] \le \frac{1}{N}\cdot[\lambda(N)N]T\Big[1 + \Big[N(1 - e^{-(\kappa_{\max}+T)/N})\Big] + \ldots + \Big[N(1 - e^{-(\kappa_{\max}+T)/N})\Big]^{R-1}\Big],$$

and from the assumptions about $\lambda(N)$, it is clear that this vanishes as $N \to \infty$.

So it remains to show that,

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{t\in[t_c,T]}\left|\int_{t_c}^{t}\sum_{r=R}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s) - \mathrm{d}\Psi^N(r,i,s)\Big]\right|\right] = 0, \qquad (5.46)$$

for a fixed value of $R \in \mathbb{N}$ to be chosen shortly.

Recall that $(\mathcal{F}^N(t))_{t\ge 0}$ is the natural filtration of the random type flow process π^N. We now define $(\bar{\mathcal{F}}^N(t))_{t\ge 0}$ to be the natural filtration of the collection of processes

$$\pi^N(\cdot) \quad \text{and} \quad \Psi^N(r,i,\cdot), \text{ for all } r \ge 0,\ i \in [k].$$

Note that, in a multitype frozen percolation process, conditional on the set of alive vertices and their types at time t, the graph structure of the frozen vertices is independent of $\mathcal{G}^N(t)$, the graph with types on alive vertices. So, although this filtration $(\bar{\mathcal{F}}^N)$ is finer than $(\mathcal{F}^N)$, Proposition 5.2 remains true after replacing conditioning on $\mathcal{F}^N(t)$ with conditioning on $\bar{\mathcal{F}}^N(t)$.

We now use some notation from Chapter 3 and Theorem 3.20. Recall that for some vertex v in some multitype graph G, we let $W_i^{\ge R}$ be the number of type i vertices in G at distance at least R from v. Now, for each $s \in [0,T]$, and $R \in \{0,\ldots,N-1\}$, conditional on $\bar{\mathcal{F}}^N(s-)$ and the event that there is a lightning strike at time s, the distribution of $\sum_{r=R}^{N-1}\Big(\Psi^N(r,s)-\Psi^N(r,s-)\Big)$ is the same as the distribution of $W^{\geq R}$ corresponding to a uniformly-chosen vertex in $G^N(N\pi^N(s-),\kappa(s))$.

We take $\tau_0 := t_c$, and let the times that lightning strikes an alive vertex after t_c be $t_c < \tau_1 < \tau_2 < \ldots$. Set $\alpha := \max\{m : \tau_m \leq T\}$ to be the number of such lightning strikes until time T. Since $\tau_1,\ldots,\tau_\alpha$ are precisely those times $t \in (t_c, T]$ for which $\pi^N(t-)-\pi^N(t) > 0$, each τ_m is an $(\mathcal{F}^N)$-stopping time, and thus an $(\bar{\mathcal{F}}^N)$-stopping time too. Now consider for $m = 0,1,\ldots,\alpha$, the discrete process

$$\begin{aligned} Y_m^N &:= \int_{t_c}^{\tau_m}\sum_{r\geq R}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s)-\mathrm{d}\Psi^N(r,i,s)\Big],\\ &= \sum_{\ell=1}^{m}\Bigg\{\mu_i(\tau_\ell,\pi^N(\tau_i-))\Bigg[\sum_{r\geq R}\Psi^N(r,\tau_\ell)-\sum_{r\geq R}\Psi^N(r,\tau_\ell-)\Bigg]\\ &\qquad -\Bigg[\sum_{r\geq R}\Psi^N(r,i,\tau_\ell)-\sum_{r\geq R}\Psi^N(r,i,\tau_\ell-)\Bigg]\Bigg\}. \end{aligned}$$

Then $(Y_m^N)_{0\leq m\leq\alpha}$ is adapted to $\Big(\bar{\mathcal{F}}^N(\tau_m)\Big)_{0\leq m\leq\alpha}$, and records the accumulation of error between the true proportion of types lost beyond radius R, and the proportion expected from the left-eigenvectors, as successive components are frozen.

We also define, for $m = 0,1,\ldots,\alpha$,

$$\xi_m^N := \int_{t_c}^{\tau_m}\sum_{r\geq R}\mathrm{d}\Psi^N(r,s) = \sum_{\ell=1}^{m}\Bigg[\sum_{r\geq R}\Psi^N(r,\tau_\ell)-\sum_{r\geq R}\Psi^N(r,\tau_\ell-)\Bigg],$$

the discrete process recording the proportion of mass lost beyond radius R after successive lightning strikes. This process $(\xi_m^N)_{0\leq m\leq\alpha}$ is also adapted to $\Big(\bar{\mathcal{F}}^N(\tau_m)\Big)_{0\leq m\leq\alpha}$.

We will now compare the increments of Y^N and the increments of ξ^N in expectation using Theorem 3.20. In particular, we will need to exclude the possibility that any component of π^N becomes too small, or that $\rho(\pi^N(t)\circ\kappa(t))$ becomes too large. Furthermore, to apply Lemma 5.18 we will have to ignore increments where the total mass lost is too large. All of these events happen with vanishing probability, and the quantities under consideration are uniformly bounded. Rather than condition that none of these events occur (which would affect the individual increments), we will exclude any pathological behaviour step-by-step for each freezing event, so as to preserve the Markov property.

Recall the definition of η from Lemma 5.14. Set $\eta' = \min(\eta, \kappa_{\min} \vee t_c) > 0$. Choose some $\delta \in (0,1)$, and consider $\epsilon = \epsilon(\delta, \eta', T + \kappa_{\max}), R = R(\delta, \eta', T + \kappa_{\max})$ as defined in Theorem 3.20. Consider the events

$$\Theta_m^{N,\eta',\epsilon} := \left\{ \pi_j^N(\tau_m -) \geq \eta', \forall j \in [k], \sup_{t \in [0,\tau_m)} \rho(\pi^N(t) \circ \kappa(t)) \leq 1 + \epsilon \right\},$$

each of which is $\mathcal{F}^N(\tau_m -)$-measurable, and thus also $\bar{\mathcal{F}}^N(\tau_m)$-measurable. On the event $\Theta_m^{N,\eta',\epsilon}$, the graphs $\mathcal{G}^N(s)$ satisfy the conditions of Theorem 3.20 for all $s \in [0, \tau_m)$. Note that

$$\Theta_1^{N,\eta',\epsilon} \supset \ldots \supset \Theta_\alpha^{N,\eta',\epsilon} \supset \Theta^{N,\eta',\epsilon} := \left\{ \pi_j^N(T) \geq \eta', \forall j \in [k], \sup_{t \in [0,T]} \rho(\pi^N(t) \circ \kappa(t)) \leq 1 + \epsilon \right\}.$$

We know from (5.28) and (5.29) that

$$\lim_{N \to \infty} \mathbb{P}\Big(\Theta^{N,\eta',\epsilon}\Big) = 1.$$

We also have $\chi = \chi(\epsilon, \eta')$ given by Theorem 3.9. We define

$$\xi_m^{N,\chi} := \sum_{\ell=1}^m \mathbb{1}_{\{\xi_\ell^N - \xi_{\ell-1}^N \leq \chi\}} \Big(\xi_\ell^N - \xi_{\ell-1}^N\Big), \quad m = 0, 1, \ldots, \alpha,$$

which counts the proportion of vertices frozen from beyond radius R, ignoring those occasions when the number of such vertices is greater than χN. (Recall that ξ^N has been rescaled like Φ^N, so that losing more than χN vertices beyond radius R corresponds to $\xi_\ell^N - \xi_{\ell-1}^N > \chi$.) Analogously, we define

$$Y_m^{N,\chi} := \sum_{\ell=1}^m \mathbb{1}_{\{\xi_\ell^N - \xi_{\ell-1}^N \leq \chi\}} \mathbb{1}_{\Theta_\ell^{N,\eta',\epsilon}} \Big(Y_\ell^N - Y_{\ell-1}^N\Big), \quad m = 0, 1, \ldots, \alpha,$$

which describes the accumulation of error in (5.35) when components of size at most χN are frozen, *and* when the graph satisfies the conditions for Theorem 3.20. Observe that $\alpha \le N$ by construction, so we also define

$$\xi_m^{N,\chi} = \xi_\alpha^{N,\chi}, \quad Y_m^{N,\chi} = Y_\alpha^{N,\chi}, \quad m = \alpha+1,\dots,N.$$

This pair of processes $(\xi^{N,\chi}, Y^{N,\chi})$ is adapted to the filtration $\mathcal{H}^N = (\mathcal{H}_m^N)_{0\le m\le N}$ defined by $\mathcal{H}_m^N := \bar{\mathcal{F}}^N(\tau_{m+1}-)$, for $m < \alpha$ and $H_m^N = \bar{\mathcal{F}}^N(\tau_\alpha)$ for $m \ge \alpha$. Observe that $\xi^{N,\chi}$ is non-decreasing and

$$\left|Y_{m+1}^{N,\chi} - Y_m^{N,\chi}\right| \le \xi_{m+1}^{N,\chi} - \xi_m^{N,\chi} \le \chi.$$

Furthermore, on $\Theta_{m+1}^{N,\eta',\epsilon}$ (which is $\mathcal{H}_m^N$-measurable),

$$\xi_{m+1}^{N,\chi} - \xi_m^{N,\chi} \,\Big|\, \mathcal{H}_m^N \overset{d}{=} W^{\ge R}\mathbb{1}_{A_\chi},$$

where $A_\chi = \{||W^{\ge R}|| \le \chi N\}$, with the IRG taken to be $G^N(N\pi^N(\tau_{m+1}-), \kappa(\tau_{m+1}))$. Similarly, again on $\Theta_{m+1}^{N,\eta',\epsilon}$,

$$Y_{m+1}^{N,\chi} - Y_m^{N,\chi} \,\Big|\, \mathcal{H}_m^N \overset{d}{=} W_i^{\ge R}\mathbb{1}_{A_\chi} - \mu_i(\tau_{m+1}, \pi^N(\tau_{m+1}-))W^{\ge R}\mathbb{1}_{A_\chi}.$$

On $(\Theta_{m+1}^{N,\eta',\epsilon})^c$, the increment $Y_{m+1}^{N,\chi} - Y_m^{N,\chi} \,\Big|\, \mathcal{H}_m^N$ is zero. Therefore, taking expectations and applying Theorem 3.20, we obtain

$$\left|\mathbb{E}\Big[Y_{m+1}^{N,\chi} - Y_m^{N,\chi} \,|\, \mathcal{H}_m^N\Big]\right| \le \delta\mathbb{E}\Big[\xi_{m+1}^{N,\chi} - \xi_m^{N,\chi} \,|\, \mathcal{H}_m^N\Big], \quad \text{a.s.}, \quad m \ge 0.$$

Thus, for $K = \lfloor\sqrt{\frac{1}{\chi}}\rfloor$, the processes $\xi^{N,\chi}$ and $Y^{N,\chi}$ precisely satisfy the conditions for Lemma 5.18. On the event $\Theta^{N,\eta,\epsilon}$,

$$\sup_{t\in[t_c,T]}\left|\int_{t_c}^t \sum_{r=R}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s) - \mathrm{d}\Psi^N(r,i,s)\Big]\right| = \sup_{0\le m\le N}\left|Y_m^N\right|.$$

Therefore, by Lemma 5.18

$$\limsup_{N\to\infty}\mathbb{E}\left[\sup_{t\in[t_c,T]}\left|\int_{t_c}^t \sum_{r=R}^{N-1}\Big[\mu_i(s,\pi^N(s-))\mathrm{d}\Psi^N(r,s) - \mathrm{d}\Psi^N(r,i,s)\Big]\right|\right]$$

$$\leq 2K^{-1} + \delta + \limsup_{N\to\infty} 2(1 - \mathbb{P}\Big(\Theta^{N,\eta',\epsilon}\Big)) = 2K^{-1} + \delta.$$

Our choice of δ was arbitrary, but as we take $\delta \to 0$, we may assume $\epsilon \to 0$ and thus, by Theorem 3.9, $\chi \to 0$ also. Hence $K \to \infty$. So (5.46) and (5.45) follow, and the proof of Lemma 5.19, and Theorem 5.7 is complete. □

5.4 Limits in time for frozen percolation type flows

It is natural to ask about the behaviour of a frozen percolation type flow as $t \to \infty$. First, we give a quick argument why $\Phi(t) \to 0$ as $t \to \infty$. From the criticality condition (5.3) and Corollary 3.11, for $t \geq t_c$,

$$1 = \rho(\kappa(t) \circ \pi(t)) \geq \rho(t\mathbf{1} \circ \pi(t)) = t\rho(\mathbf{1} \circ \pi(t)).$$

Therefore $\rho(\mathbf{1} \circ \pi(t)) \leq 1/t$. But note that $(1, \ldots, 1)^T$ is a right-eigenvector of $\mathbf{1} \circ \pi(t)$, with eigenvalue $\Phi(t)$. Therefore

$$\Phi(t) \leq 1/t. \tag{5.47}$$

Now we prove that the proportion of types among the alive vertices converges as $t \to \infty$.

Proposition 5.20. For any frozen percolation type flow π, $\lim_{t\to\infty} \frac{\pi(t)}{\Phi(t)}$ exists and is positive.

Proof. Directly from (5.4), $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -\phi(t)$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi(t)}{\Phi(t)}\right) \overset{(5.4)}{=} \frac{\phi(t)}{\Phi(t)}\left(\frac{\pi(t)}{\Phi(t)} - \mu((\kappa + t\mathbf{1}) \circ \pi(t))\right).$$

Note that $\frac{\pi(t)}{\Phi(t)} = \mu(t\mathbf{1} \circ \pi(t))$, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi(t)}{\Phi(t)}\right) = \frac{\phi(t)}{\Phi(t)}\Big[\mu(t\mathbf{1} \circ \pi(t)) - \mu((\kappa + t\mathbf{1}) \circ \pi(t))\Big]. \tag{5.48}$$

Consider the sets of positive matrices

$$\mathcal{A} := \{t\mathbf{1} \circ \pi(t) : t \geq t_c\}, \quad \mathcal{B} := \{(\kappa + t\mathbf{1}) \circ \pi(t) : t \geq t_c\}.$$

Now, for any $A \in \mathcal{A} \cup \mathcal{B}$,

$$A_{i,j} \leq (\kappa_{\max} + t)\pi_j(t) \leq (\kappa_{\max} + t)\Phi(t) \leq \frac{\kappa_{\max}}{t_c} + 1,$$

where the final inequality follows from (5.47). Hence matrices in $\mathcal{A} \cup \mathcal{B}$ are bounded in $\mathbb{R}_{\geq 0}^{k \times k}$ and thus the closure $\overline{\mathcal{A} \cup \mathcal{B}}$ is compact. Any matrix in $\overline{\mathcal{A} \cup \mathcal{B}}$ has the property that any row has either all positive entries, or all zero entries, and at least one row has all positive entries. Thus the Perron root of any matrix in $\overline{\mathcal{A} \cup \mathcal{B}}$ is a simple eigenvalue, and Lemma 3.15 applies, with $\mathbb{A} = \overline{\mathcal{A} \cup \mathcal{B}}$. In particular, there exists a constant $C = C(\mathbb{A})$ (depending on κ and $\pi(0)$) such that

$$||\mu(A) - \mu(B)||_1 \leq C \max_{i,j \in [k]} |A_{i,j} - B_{i,j}|, \quad A, B \in \overline{\mathcal{A} \cup \mathcal{B}}.$$

So from (5.48),

$$\left|\left| \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\pi(t)}{\Phi(t)} \right) \right|\right|_1 \leq C \cdot \frac{\phi(t)}{\Phi(t)} \cdot \max_{i,j} \kappa_{i,j}\pi_j(t) \leq C\phi(t)\kappa_{\max}.$$

Therefore, if we write $g(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi(t)}{\Phi(t)}\right)$, we have

$$\int_{t_c}^{\infty} ||g(t)||_1 \mathrm{d}t \leq C\kappa_{\max} \int_{t_c}^{\infty} \phi(t)\mathrm{d}t \leq C\kappa_{\max}\Phi(0) < \infty,$$

and it follows that $\frac{\pi(t)}{\Phi(t)}$ converges as $t \to \infty$.

We now show that the limit is positive. For this, we will use a similar argument to the proof of Lemma 5.14, but now using the statement of Theorem 5.7 to give stronger bounds involving Φ.

Recall that $\pi(0)$ and κ are fixed. Now, for each $N \in \mathbb{N}$, we take N IID samples from $\pi(0)$, and let $p^N \in \mathbb{N}_0^k$ be the vector recording the number of occurences of each type. Clearly, by WLLN $p^N/N \xrightarrow{d} \pi(0)$ as $N \to \infty$. We will consider coupling frozen percolation processes with initial types given by p^N, as N varies.

Fix some sequence $(\lambda(N))$ satisfying the critical scaling $1/N \ll \lambda(N) \ll 1$. Observe that there is a natural coupling between the processes $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}$ and $\mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}$ under which the restriction of the latter to vertex set $[N]$ is equal to the former until the first time an edge is added between $N+1$ and an alive vertex in $[N]$. (This time might be zero, if there is such an edge in the initial graph $\mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}$.) We fix a time $T > 0$. Theorem 5.7 applies to both sequences of processes $(\mathcal{G}^{N,p^N,\kappa,\lambda(N)})$ and $(\mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)})$, since certainly $\lambda(N-1)$ also satisfies the critical scaling. While this theorem is stated in terms of convergence in probability, it also holds in expectation since the processes π^N are uniformly bounded in $\mathbb{R}^k$. Thus, for each $i \in [k]$,

$$\begin{aligned}\pi_i(T) &= \lim_{N\to\infty} \mathbb{E}\left[\pi_i^{N+1}(T)\right] \\ &= \lim_{N\to\infty} \mathbb{P}\Big(\text{type}(N+1) = i,\ N+1 \text{ alive in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(T)\Big).\end{aligned}$$

Although it leads to a weaker bound, it is more convenient to consider the probability that vertex $N+1$ is both alive *and isolated* in $\mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(T)$. This event is particularly tractable under the coupling proposed above. For, as long as $N+1$ is isolated, an edge forms between $N+1$ and $[N]$ at rate $\frac{1}{N}\#\{\text{alive vertices in } [N]\}$. So, if $\Phi^N(t)$ remains the proportion of alive vertices in $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}$, we can control the probability that $N+1$ remains isolated in $\mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}$ *conditional* on the evolution of $\mathcal{G}^{N,p^N,\kappa,\lambda(N)}$. That is,

$$\begin{aligned}&\mathbb{P}\Big(N+1 \text{ alive and isolated in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(T) \,\Big|\, \mathcal{G}^{N,p^N,\kappa,\lambda(N)}\big|_{[0,T]}\Big) \\ &\quad = \mathbb{P}\Big(N+1 \text{ isolated in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(0)\Big) \\ &\qquad \times \mathbb{P}(N+1 \text{ not struck by lightning on } [0,T]) \times \exp\left(-\int_0^T \Phi^N(s)\mathrm{d}s\right).\end{aligned}$$

Since the second and third probabilities are independent of the type of $N+1$, we can include this in the calculation. Then,

$$\begin{aligned}
&\mathbb{P}\Big(\text{type}(N+1)=i,\ N+1 \text{ alive and isolated in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(T)\ \Big|\ \mathcal{G}^{N,p^N,\kappa,\lambda(N)}\big|_{[0,T]}\Big)\\
&\quad= \mathbb{P}\Big(\text{type}(N+1)=i, N+1 \text{ isolated in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(0)\Big)\\
&\qquad\times \mathbb{P}(N+1 \text{ not struck by lightning on } [0,T]) \times \exp\bigg(-\int_0^T \Phi^N(s)\mathrm{d}s\bigg). \qquad (5.49)
\end{aligned}$$

Only the third of these terms is random. We now consider its expectation. Note that the map $f \mapsto \exp\Big(-\int_0^T f(s)\mathrm{d}s\Big)$ from $C_b([0,T])$ to $\mathbb{R}$ is continuous with respect to the uniform topology on $[0,T]$. Since $\Phi^N \xrightarrow{d} \Phi$ uniformly on $[0,T]$, it follows that

$$\lim_{N\to\infty} \mathbb{E}\bigg[\exp\bigg(-\int_0^T \Phi^N(s)\mathrm{d}s\bigg)\bigg] = \exp\bigg(-\int_0^T \Phi(s)\mathrm{d}s\bigg) \overset{(5.47)}{\geq} \exp\bigg(-1-\int_1^T \frac{\mathrm{d}s}{s}\bigg) = \frac{1}{Te}.$$

So, from (5.49) and the law of total probability,

$$\begin{aligned}
\pi_i(T) &\geq \limsup_{N\to\infty} \mathbb{P}\Big(\text{type}(N+1)=i,\ N+1 \text{ alive and isolated in } \mathcal{G}^{N+1,p^{N+1},\kappa,\lambda(N)}(T)\Big)\\
&\geq \bigg[\lim_{N\to\infty} \frac{p_i^{N+1}}{N+1} e^{-\kappa_{\max}}\bigg]\bigg[\lim_{N\to\infty} e^{-\lambda(N+1)T}\bigg] \lim_{N\to\infty} \mathbb{E}\bigg[\exp\bigg(-\int_0^T \Phi^N(s)\mathrm{d}s\bigg)\bigg]\\
&\geq \pi_i(0)e^{-\kappa_{\max}} \cdot \frac{1}{Te}. \qquad (5.50)
\end{aligned}$$

Combining (5.47) and (5.50), we obtain

$$\frac{\pi_i(T)}{\Phi(T)} \geq \frac{\pi_i(0)e^{-(\kappa_{\max}+1)}/T}{1/T} = \pi_i(0)e^{-(\kappa_{\max}+1)},$$

and thus $\lim_{T\to\infty} \frac{\pi(T)}{\Phi(T)}$ has positive components. □

Bibliography

[1] Addario-Berry, L., Broutin, N., and Goldschmidt, C. The continuum limit of critical random graphs. *Probab. Theory Related Fields 152*, 3-4 (2012), 367–406.

[2] Albert, R., and Barabási, A.-L. Statistical mechanics of complex networks. *Rev. Mod. Phys*, 2002.

[3] Aldous, D. The continuum random tree II: an overview. In *Stochastic Analysis* (1990), Cambridge Univ. Press, pp. 23–70.

[4] Aldous, D. The continuum random tree. III. *Ann. Probab. 21*, 1 (1993), 248–289.

[5] Aldous, D. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab. 25*, 2 (1997), 812–854.

[6] Aldous, D., and Pitman, J. The standard additive coalescent. *Ann. Probab. 26*, 4 (10 1998), 1703–1726.

[7] Aldous, D. J. Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. *Bernoulli 5*, 1 (1999), 3–48.

[8] Aldous, D. J. The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc 128* (1999), 465–477.

[9] Athreya, K. B., and Ney, P. E. *Branching processes*. Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.

[10] Bak, P., Tang, C., and Wiesenfeld, K. Self-organized criticality: An explanation of the $1/f$ noise. *Phys. Rev. Lett. 59* (Jul 1987), 381–384.

[11] Bak, P., Tang, C., and Wiesenfeld, K. Self-organized criticality. *Phys. Rev. A (3) 38*, 1 (1988), 364–374.

[12] Benjamini, I., and Schramm, O. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab. 6* (2001), no. 23, 13 pp. (electronic).

[13] Bollobás, B. The evolution of random graphs. *Trans. Amer. Math. Soc. 286*, 1 (1984), 257–274.

[14] Bollobás, B. *Random graphs*, second ed., vol. 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001.

[15] Bollobás, B., Janson, S., and Riordan, O. The phase transition in inhomogeneous random graphs. *Random Structures Algorithms 31*, 1 (2007), 3–122.

[16] Britikov, V. E. Asymptotics of the number of forests made up of nonrooted trees. *Mat. Zametki 43*, 5 (1988), 672–684, 703.

[17] Collatz, L. Einschliessungssatz für die charakteristischen Zahlen von Matrizen. *Math. Z. 48* (1942), 221–226.

[18] Crane, E., Freeman, N., and Tóth, B. Cluster growth in the mean field forest fire model. *ArXiv e-prints* (May 2014).

[19] Durrett, R. *Stochastic calculus*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996. A practical introduction.

[20] Dwass, M. The total progeny in a branching process and a related random walk. *J. Appl. Probability 6* (1969), 682–686.

[21] Erdős, P. Graph theory and probability. *Canad. J. Math. 11* (1959), 34–38.

[22] Erdős, P., and Rényi, A. On random graphs. I. *Publ. Math. Debrecen 6* (1959), 290–297.

[23] Erdős, P., and Rényi, A. On the evolution of random graphs. *Bull. Inst. Internat. Statist. 38* (1961), 343–347.

[24] Ethier, S. N., and Kurtz, T. G. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[25] Evans, S. N., and Pitman, J. Construction of Markovian coalescents. *Ann. Inst. H. Poincaré Probab. Statist. 34*, 3 (1998), 339–383.

[26] Flory, P. *Principles of Polymer Chemistry*. Baker lectures 1948. Cornell University Press, 1953.

[27] Fournier, N., and Giet, J.-S. Convergence of the Marcus-Lushnikov process. *Methodol. Comput. Appl. Probab. 6*, 2 (2004), 219–231.

[28] Fournier, N., and Laurençot, P. Marcus-Lushnikov processes, Smoluchowski's and Flory's models. *Stochastic Processes and their Applications 119*, 1 (2009), 167–189.

[29] Frobenius, G. Über Matrizen aus nicht negativen Elementen. *Berl. Ber. 1912* (1912), 456–477.

[30] Gilbert, E. N. Random graphs. *Ann. Math. Statist. 30* (1959), 1141–1144.

[31] Harris, T. E. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc. 56* (1960), 13–20.

[32] Janson, S. Probability asymptotics: notes on notation. *ArXiv e-prints* (Aug. 2011). `http://arxiv.org/abs/1108.3924`.

[33] Janson, S., and Riordan, O. Susceptibility in inhomogeneous random graphs. *Electron. J. Combin. 19*, 1 (2012), Paper 31, 59.

[34] Janson, S., and Spencer, J. A point process describing the component sizes in the critical window of the random graph evolution. *Combin. Probab. Comput. 16*, 4 (2007), 631–658.

[35] Jeon, I. Existence of gelling solutions for coagulation-fragmentation equations. *Comm. Math. Phys. 194*, 3 (1998), 541–567.

[36] Joffe, A., and Spitzer, F. On multitype branching processes with $\rho \leq 1$. *J. Math. Anal. Appl. 19* (1967), 409–430.

[37] Kaigh, W. D. An invariance principle for random walk conditioned by a late return to zero. *Ann. Probability 4*, 1 (1976), 115–121.

[38] Kesten, H. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist. 22*, 4 (1986), 425–487.

[39] Kingman, J. F. C. On the genealogy of large populations. *J. Appl. Probab.*, Special Vol. 19A (1982), 27–43. Essays in statistical science.

[40] Kokholm, N. J. On Smoluchowski's coagulation equation. *J. Phys. A 21*, 3 (1988), 839–842.

[41] Laurençot, P. On a class of continuous coagulation-fragmentation equations. *J. Differential Equations 167*, 2 (2000), 245–274.

[42] Le Gall, J.-F. Random trees and applications. *Probab. Surv. 2* (2005), 245–311.

[43] Le Gall, J.-F. Random real trees. *Ann. Fac. Sci. Toulouse Math. (6) 15*, 1 (2006), 35–62.

[44] Łuczak, T. Component behavior near the critical point of the random graph process. *Random Structures Algorithms 1*, 3 (1990), 287–310.

[45] Łuczak, T., and Pittel, B. Components of random forests. *Combin. Probab. Comput. 1*, 1 (1992), 35–52.

[46] Lushnikov, A. Evolution of coagulating systems. *Journal of Colloid and Interface Science 45*, 3 (1973), 549 – 556.

[47] Lushnikov, A. Coagulation in finite systems. *Journal of Colloid and Interface Science 65*, 2 (1978), 276 – 285.

[48] Magnus, J. R., and Neudecker, H. *Matrix differential calculus with applications in statistics and econometrics.* Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 1988.

[49] Marcus, A. H. Stochastic coalescence. *Technometrics 10* (1968), 133–143.

[50] McLeod, J. B. On an infinite set of non-linear differential equations. *Quart. J. Math. Oxford Ser. (2) 13* (1962), 119–128.

[51] Merle, M., and Normand, R. Self-organized criticality in a discrete model for Smoluchowski's equation. *ArXiv e-prints* (Oct. 2014).

[52] Mode, C. J. *Multitype branching processes. Theory and applications.* Modern Analytic and Computational Methods in Science and Mathematics, No. 34. American Elsevier Publishing Co., Inc., New York, 1971.

[53] Normand, R., and Zambotti, L. Uniqueness of post-gelation solutions of a class of coagulation equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 28*, 2 (2011), 189 – 215.

[54] Norris, J. R. Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab. 9*, 1 (1999), 78–109.

[55] Norris, J. R. Cluster coagulation. *Comm. Math. Phys. 209*, 2 (2000), 407–435.

[56] O'Connell, N. Some large deviation results for sparse random graphs. *Probab. Theory Related Fields 110*, 3 (1998), 277–285.

[57] Perron, O. Zur Theorie der Matrices. *Math. Ann. 64*, 2 (1907), 248–263.

[58] Pitman, J. Coalescent random forests. *J. Combin. Theory Ser. A 85*, 2 (1999), 165–193.

[59] Pitman, J. *Combinatorial stochastic processes*, vol. 1875 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.

[60] Ráth, B. Mean field frozen percolation. *J. Stat. Phys. 137*, 3 (2009), 459–499.

[61] Ráth, B., and Tóth, B. Erdős-Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab. 14* (2009), no. 45, 1290–1327.

[62] Revuz, D., and Yor, M. *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1991.

[63] Smoluchowski, M. V. Drei Vortrage uber Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen. *Zeitschrift fur Physik 17* (1916), 557–585.

[64] Söderberg, B. General formalism for inhomogeneous random graphs. *Phys. Rev. E (3) 66*, 6 (2002), 066121, 6.

[65] Strassen, V. The existence of probability measures with given marginals. *Ann. Math. Statist. 36* (1965), 423–439.

[66] Stroock, D. W., and Varadhan, S. R. S. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math. 24* (1971), 147–225.

[67] Stroock, D. W., and Varadhan, S. R. S. *Multidimensional diffusion processes.* Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[68] van der Hofstad, R. Random graphs and complex networks. In *In preparation* (2008).

[69] van der Hofstad, R., and Keane, M. An elementary proof of the hitting time theorem. *Amer. Math. Monthly 115*, 8 (2008), 753–756.

[70] Wielandt, H. Unzerlegbare, nicht negative Matrizen. *Math. Z. 52* (1950), 642–648.

[71] Williams, D. *Probability with martingales.* Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

[72] Zedek, M. Continuity and location of zeros of linear combinations of polynomials. *Proc. Amer. Math. Soc. 16* (1965), 78–84.

[73] Zolotarev, V. M. *One-dimensional stable distributions*, vol. 65 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.